Dirac Notation, Bloch Sphere, and Quantum Gates

From Single-Qubit States to Bell and GHZ States

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Dirac Notation: Kets and Bras

Qubit-ordering convention

Throughout these slides, multi-qubit states are written with the highest-index qubit on the left. For three qubits, $|q_2 q_1 q_0\rangle$; thus q_0 is the rightmost bit.

Ket notation represents a quantum state as a column vector:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

A general single-qubit pure state is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

with

$$|\alpha|^2 + |\beta|^2 = 1.$$

Bra notation is the conjugate transpose:

$$\langle\psi| = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}.$$

The inner product is

$$\langle\phi|\psi\rangle,$$

and the normalization condition is

$$\langle\psi|\psi\rangle = 1.$$

Multiple Qubits and Tensor Products

For two qubits, the computational basis states are tensor products: In this presentation, we use the Qiskit-style ordering $|q_1 q_0\rangle$ (and $|q_2 q_1 q_0\rangle$ for three qubits).

$$|ab\rangle = |a\rangle \otimes |b\rangle, \qquad a, b \in \{0, 1\}.$$

For example,

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus a general two-qubit state is

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle,$$

where $\sum_{a,b} |\alpha_{ab}|^2 = 1$.

Bloch Sphere Representation

Ignoring an unobservable global phase, every single-qubit pure state can be written as

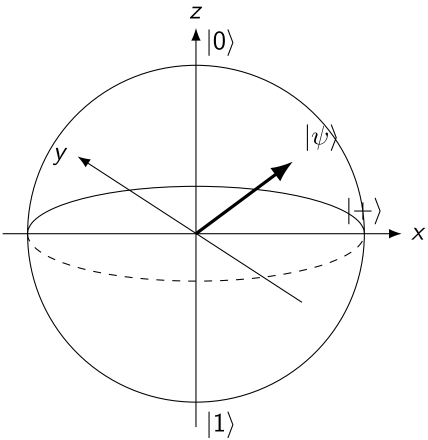
$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle.$$

The corresponding Bloch vector is

$$\boldsymbol{r} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta).$$

Important points:

$$|0\rangle : +z, \quad |1\rangle : -z, \quad |+\rangle : +x, \quad |-\rangle : -x.$$



Hadamard Gate: Matrix Representation

The Hadamard gate is

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Its action on the computational basis is

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle,$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle.$$

Therefore, H maps the Z-basis states to the X-basis states. On the Bloch sphere, it interchanges the x and z axes (with a sign reversal of y).

Controlled-X (CX/CNOT) Gate

For the ordering $|q_1 q_0\rangle$, let q_0 be the **control** and q_1 be the **target**. The CX operation is

$$|q_1 q_0\rangle \longmapsto |q_1 \oplus q_0,\ q_0\rangle.$$

Using the basis order $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$,

$$CX = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Hence

$$|00\rangle \to |00\rangle, \quad |01\rangle \to |11\rangle, \quad |10\rangle \to |10\rangle, \quad |11\rangle \to |01\rangle.$$

Bell-State Circuit

Start from

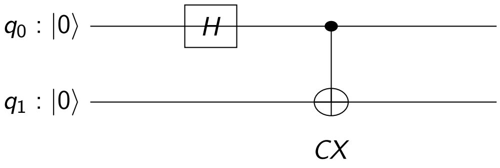
$$|\psi_0\rangle = |00\rangle.$$

Apply H to qubit 0, then apply $CX_{0\to1}$:

$$|00\rangle \xrightarrow{I\otimes H} \frac{|00\rangle + |01\rangle}{\sqrt{2}} \xrightarrow{CX} \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

The final state is the Bell state

$$\boxed{|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}}.$$



The state cannot be factored into two independent single-qubit states: it is **entangled**.

Bell State: Matrix Derivation

Using

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

first apply H to q_0, i.e., $I \otimes H$:

$$(I \otimes H)|00\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$CX \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

From Bell State to a Three-Qubit GHZ State

Start with three qubits:

$$|\psi_0\rangle = |000\rangle.$$

Apply a Hadamard gate to qubit 0:

$$|\psi_1\rangle = (I \otimes I \otimes H)|000\rangle = \frac{|000\rangle + |001\rangle}{\sqrt{2}}.$$

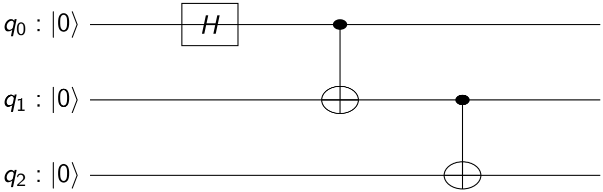
Apply $CX_{0\to1}$:

$$|\psi_2\rangle = \frac{|000\rangle + |011\rangle}{\sqrt{2}}.$$

Finally apply $CX_{1\to2}$:

$$\boxed{|\psi_3\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}} = |GHZ_3\rangle.}$$

Three-Qubit GHZ Circuit



$$|000\rangle \xrightarrow{H_0} \frac{|000\rangle + |001\rangle}{\sqrt{2}} \xrightarrow{CX_{0\to 1}} \frac{|000\rangle + |011\rangle}{\sqrt{2}} \xrightarrow{CX_{1\to 2}} \frac{|000\rangle + |111\rangle}{\sqrt{2}}.$$

Generalization to an n-Qubit GHZ State

The same construction generalizes naturally:

$$|0\rangle^{\otimes n} \xrightarrow{H_0} \frac{|0\cdots0\rangle + |0\cdots01\rangle}{\sqrt{2}}.$$

A chain of CX gates propagates the correlation:

$$CX_{0\to1},\ CX_{1\to2},\ \ldots,\ CX_{n-2\to n-1}.$$

The final state is

$$\boxed{|GHZ_n\rangle = \frac{|0\rangle^{\otimes n} + |1\rangle^{\otimes n}}{\sqrt{2}}.}$$

For $n = 2$, this is exactly the Bell state $|\Phi^+\rangle$.

Key Takeaways

- Dirac notation provides a compact language for quantum states and operations.
- A single-qubit pure state can be visualized geometrically on the Bloch sphere.
- The Hadamard gate creates equal-amplitude superpositions from computational-basis states.
- The CX gate conditionally flips a target qubit and is essential for generating entanglement.
- H followed by CX transforms $|00\rangle$ into the Bell state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

- Extending the CX correlation to a third qubit produces

$$|GHZ_3\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}.$$

Thank You

Questions?