Optimization

Homework 3 Solutions

1. Minimize $-2x_1-x_2$

Subject to $x_1+x_3=2$

$x_1+x_2+x_4=3$

$x_1+2x_2+x_5=5$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

2. .

$$\begin{bmatrix}2 & -1 & 2 & -1 & 3\\ 1 & 2 & 3 & 1 & 0\\ 1 & 0 & -2 & 0 & -5\end{bmatrix}[x_1, x_2, x_3, x_4, x_5]^T=\begin{bmatrix}14\\ 5\\ -10\end{bmatrix}.$$

The total number of possible basic solution is at most 10.

Case1：$x_1=x_2=0 \rightarrow x=[0,0,\frac{65}{19},\frac{-100}{19},\frac{12}{19}]^T$

Case2：$x_1=x_3=0 \rightarrow x=[0,13,0,-21,2]^T$

Case3：$x_1=x_4=0 \rightarrow x=[0,-\frac{100}{23},\frac{105}{23},0,\frac{4}{23}]^T$

Case4：$x_1=x_5=0 \rightarrow x=[0,-6,5,2,0]^T$

Case5：$x_3=x_2=0 \rightarrow x=[\frac{65}{18},0,0,\frac{25}{18},\frac{49}{18}]^T$

Case6：$x_4=x_2=0 \rightarrow x=[\frac{20}{7},0,\frac{5}{7},0,\frac{16}{7}]^T$

Case7：$x_5=x_2=0 \rightarrow x=[-\frac{12}{11},0,\frac{49}{11},-\frac{80}{11},0]^T$

Case8：$x_3=x_4=0 \rightarrow x=[\frac{105}{31},\frac{25}{31},0,0,\frac{83}{31}]^T$

Case9：$x_3=x_5=0 \rightarrow x=[-10,49,0,-83,0]^T$

Case10：$x_4=x_5=0 \rightarrow x=[-\frac{4}{17},\frac{80}{17},\frac{83}{17},0,0]^T$

3.

(a) $A=\begin{bmatrix}3 & 1 & 0 & 1\\ 6 & 2 & 1 & 1\end{bmatrix}$, $b=\begin{bmatrix}4\\ 5\end{bmatrix}$, $c=\begin{bmatrix}2\\ -1\\ -1\end{bmatrix}$

(b) $\begin{bmatrix}3 & 1 & 0 & 1 & 4\\ 3 & 1 & 1 & 0 & 1\\ 5 & 0 & 0 & 0 & 1\end{bmatrix}$

(c) $x=[0,0,1,4]^T$

objective value $c^Tx = -1$

(d) $[r_1,r_2,r_3,r_4]=[5,0,0,0]$

(e) Yes, because no reduced cost coefficient is negative.

(f) According to Proposition 16.1, artificial problem has optimal feasible solution, so this problem has basic feasible solution.

(g) $\begin{bmatrix} 0 & 0 & -1 & 1 & 3 \\ 1 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 2 & -1 & -1 & 0 & 0 \end{bmatrix}$

4. Minimize $-x_1 - x_2 - 3x_3$

Subject to $x_1 + x_3 = 1$

$x_2 + x_3 = 2$

$x_1, x_2, x_3 \geq 0$

a_1	a_2	a_3	b
1	0	1	1
0	1	1	2

➡

a_1	a_2	a_3	b
1	0	1	1
-1	1	0	1

$x=[1,2,0]^T$ $\quad$ $x=[0,1,1]^T$

$r_3 = -3-[1,1][1,1]^T = -1 < 0$ $\quad$ $r_1 = -1-[1,-1][-3,-1]^T = 1 > 0$

Hence, $x=[0,1,1]^T$ is optimal basic feasible solution, objective function value= 4.

5. Minimize $-2x_1 - x_2$

Subject to $x_1 + x_3 = 5$

$x_2 + x_4 = 7$

$x_1 + x_2 + x_5 = 9$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

①

a_1	a_2	a_3	a_4	a_5	b
1	0	1	0	0	5
0	1	0	1	0	7
1	1	0	0	1	9

➡

②

a_1	a_2	a_3	a_4	a_5	b
1	0	1	0	0	5
0	1	0	1	0	7
1	0	0	-1	1	2

$x=[0,0,5,7,9]^T$ $\quad$ $x=[0,0,5,7,9]^T$

$r_1 = -2-[1,0,1][0,0,0]^T = -2 < 0$ $\quad$ $r_1 = -2-[1,0,1][0,-1,0]^T = -2 < 0$

$r_3 = -1 < 0$ $\quad$ $r_4 = 0-[0,1,-1][0,-1,0]^T = 1 > 0$

③

a_1	a_2	a_3	a_4	a_5	b
0	0	1	1	-1	3
0	1	0	1	0	7
1	0	0	-1	1	2

➡

④

a_1	a_2	a_3	a_4	a_5	b
0	0	1	1	-1	3
0	1	-1	0	1	4
1	0	1	0	0	5

r_4= 0-[1,1,-1][0,-1,-2]^T =-1< 0 r_3= 0-[1,-1,1][0,-1,-2]^T = 1> 0

r_5= 0-[-1,0,1][0,-1,-2]^T =2 >0 r_5= 0-[-1,1,0][0,-1,-2]^T =1>0

Hence, x=[5,4,0,3,0]^T is optimal basic feasible solution, objective function value= 14.

6. If all y_{iq}<0, then each value in the vector$[y_{10} - \varepsilon y_{1q}, y_{20} - \varepsilon y_{2q}, \ldots, y_{m0} - \varepsilon y_{mq}, 0, \ldots \varepsilon, \ldots 0]^T$ will increase as ε is increased, hence it will make this problem is unbounded.

7. Minimize $4x_1+3x_2$

Subject to $5x_1+x_2 \geq 11$

$2x_1+x_2 \geq 8$

$x_1+2x_2 \geq 7$

$x_1,x_2 \geq 0$

Dual ➡

Minimize $11\lambda_1+8\lambda_2+7\lambda_3$

Subject to $5\lambda_1+2\lambda_2+\lambda_3 \leq 4$

$\lambda_1+\lambda_2+2\lambda_3 \leq 3$

①

	a_1	a_2	a_3	a_4	a_5	b
	5	2	1	1	0	4
	1	1	2	0	1	3
c^T	-11	-8	-7	0	0	0

➡

②

	a_1	a_2	a_3	a_4	a_5	b
	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0	$\frac{4}{5}$
	0	$\frac{3}{5}$	$\frac{9}{5}$	$-\frac{1}{5}$	1	$\frac{11}{5}$
	0	$-\frac{18}{5}$	$-\frac{24}{5}$	$\frac{11}{5}$	0	$\frac{44}{5}$

③

	a_1	a_2	a_3	a_4	a_5	b
	1	$\frac{1}{3}$	0	$\frac{2}{9}$	$-\frac{1}{9}$	$\frac{5}{9}$
	0	$\frac{1}{3}$	1	$-\frac{1}{9}$	$\frac{5}{9}$	$\frac{11}{9}$
c^T	0	-10	0	$\frac{25}{3}$	$\frac{40}{3}$	$\frac{220}{9}$

➡

④

	a_1	a_2	a_3	a_4	a_5	b
	3	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$
	-1	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
	6	0	0	3	2	18

$\lambda=[0, \frac{5}{3}, \frac{2}{3}, 0, 0]^T$ is optimal solution.

8. .

(a). Minimize 0

Subject to $A^T\lambda \geq c^T$

$\lambda \geq 0$

(b). From dual problem, minimum is zero. Because the maximum of primal problem must be equal to the minimum of dual problem. The maximum of primal problem is zero as well.

(c). From the primal problem, the constraint is bounded which the condition is that $x=0$. But from the dual problem, the constraint also must be bounded which the condition is that A must be full rank.