7.3 FINDING THE EIGENVECTORS OF A MATRIX

After we have found an eigenvalue $\lambda$ of an $n \times n$ matrix $A$, we have to find the vectors $\vec{v}$ in $\mathbb{R}^n$ such that

$$A\vec{v} = \lambda \vec{v} \text{ or } (\lambda I_n - A)\vec{v} = \vec{0}$$

In other words, we have to find the kernel of the matrix $\lambda I_n - A$.

**Definition 7.3.1 Eigenspace**

Consider an eigenvalue $\lambda$ of an $n \times n$ matrix $A$. Then the kernel of the matrix $\lambda I_n - A$ is called the eigenspace associated with $\lambda$, denoted by $E_\lambda$:

$$E_\lambda = ker(\lambda I_n - A)$$

Note that $E_\lambda$ consists of all solutions $\vec{v}$ of the linear system

$$A\vec{v} = \lambda \vec{v}$$
EXAMPLE 1 Let $T(\vec{x}) = A\vec{v}$ be the orthogonal projection onto a plane $E$ in $\mathbb{R}^3$. Describe the eigenspaces geometrically.

Solution See Figure 1.
The nonzero vectors $\vec{v}$ in $E$ are eigenvectors with eigenvalue 1. Therefore, the eigenspace $E_1$ is just the plane $E$.

Likewise, $E_0$ is simply the kernel of $A$ ($A\vec{v} = \vec{0}$); that is, the line $E^\perp$ perpendicular to $E$. 
EXAMPLE 2 Find the eigenvectors of the matrix \( A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \).

Solution
See Section 7.2, Example 1, we saw the eigenvalues are 5 and -1. Then

\[
E_5 = \ker(5I_2 - A) = \ker \begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix} = \ker \begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

\[
E_{-1} = \ker(-I_2 - A) = \ker \begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix} = \text{span} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

Both eigenspaces are lines, See Figure 2.
EXAMPLE 3 Find the eigenvectors of

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}.
\]

Solution

Since

\[
f_A(\lambda) = \lambda(\lambda - 1)^2
\]

the eigenvalues are 1 and 0 with algebraic multiplicities 2 and 1.

\[
E_1 = \text{ker} \begin{bmatrix}
0 & -1 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

To find this kernel, apply Gauss-Jordan Elimination:

\[
\begin{bmatrix}
0 & -1 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

The general solution of the system

\[
\begin{align*}
x_2 & = 0 \\
x_3 & = 0
\end{align*}
\]
is
\[
\begin{bmatrix}
  x_1 \\
  0 \\
  0
\end{bmatrix} = x_1 \begin{bmatrix}
  1 \\
  0 \\
  0
\end{bmatrix}
\]

Therefore,
\[
E_1 = \text{span} \begin{bmatrix}
  1 \\
  0 \\
  0
\end{bmatrix}
\]

Likewise, compute the \( E_0 \):
\[
E_0 = \text{span} \begin{bmatrix}
  1 \\
  -1 \\
  0
\end{bmatrix}
\]

Both eigenspaces are lines in the \( x_1-x_2 \)-plane, as shown in Figure 3.

Compare with Example 1. There, too, we have two eigenvalues 1 and 0, but one of the eigenspace, \( E_1 \), is a plane.
Definition 7.3.2 Geometric multiplicity
Consider an eigenvalue $\lambda$ if a matrix $A$. Then the dimension of eigenvalue $E_\lambda = \ker(\lambda I_n - A)$ is called the geometric multiplicity of eigenvalue $\lambda$. Thus, the geometric multiplicity of $\lambda$ is the nullity of matrix $\lambda I_n - A$.

Example 3 shows that the geometric multiplicity of an eigenvalue may be different from the algebraic multiplicity. We have

(algebraic multiplicity of eigenvalue $1$) = $2$,

but

(geometric multiplicity of eigenvalue $1$) = $1$.

Fact 7.3.3
Consider an eigenvalue $\lambda$ of a matrix $A$. Then

(geometric multiplicity of $\lambda$) $\leq$ (algebraic multiplicity of $\lambda$).
EXAMPLE 4 Consider an upper triangular matrix of the form

\[
A = \begin{bmatrix}
1 & \star & \star & \star & \star \\
0 & 2 & \star & \star & \star \\
0 & 0 & 4 & \star & \star \\
0 & 0 & 0 & 4 & \star \\
0 & 0 & 0 & 0 & 4
\end{bmatrix}.
\]

What can you say about the geometric multiplicity of the eigenvalue 4?

Solution

\[
E_4 = \begin{bmatrix}
3 & \star & \star & \star & \star \\
0 & 2 & \star & \star & \star \\
0 & 0 & 0 & \star & \star \\
0 & 0 & 0 & 0 & \star \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}
1 & \star & \star & \star & \star \\
0 & 1 & \star & \star & \star \\
0 & 0 & 0 & \# & \star \\
0 & 0 & 0 & 0 & \# \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The bullets on row 3 and 4 could be leading 1’s. Therefore, the rank of this matrix will be between 2 and 4, and its nullity will be between 3 and 1. We can conclude that the geometric multiplicity of the eigenvalue 4 is less than the algebraic multiplicity.
Recall Fact 7.1.3, such a basis deserves a name.

**Definition 7.3.4 Eigenbasis**
Consider an \( n \times n \) matrix \( A \). A basis of \( \mathbb{R}^n \) consisting of eigenvectors of \( A \) is called an *eigenbasis* for \( A \).

**Example 1 Revisited:** Projection on a plane \( E \) in \( \mathbb{R}^3 \). Pick a basis \( \vec{v}_1, \vec{v}_2 \) of \( E \) and a nonzero \( \vec{v}_3 \) in \( E^\perp \). The vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) form an eigenbasis. See Figure 4.

**Example 2 Revisited:** \( A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \).

The vectors \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) form an eigenbasis for \( A \), see Figure 5.

**Example 3 Revisited:** \( A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \).

There are not enough eigenvectors to form an eigenbasis. See Figure 6.
EXAMPLE 5 Consider a $3 \times 3$ matrix $A$ with three eigenvalues, 1, 2, and 3. Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be corresponding eigenvectors. Are vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ necessarily linearly independent?

Solution See Figure 7.
Consider the plane $E$ spanned by $\vec{v}_1$, and $\vec{v}_2$. We have to examine $\vec{v}_3$ can not be contained in this plane.

Consider a vector $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ in $E$ (with $c_1 \neq 0$ and $c_2 \neq 0$). Then $A\vec{x} = c_1A\vec{v}_1 + c_2A\vec{v}_2 = c_1\vec{v}_1 + 2c_2\vec{v}_2$. This vector can not be a scalar multiple of $\vec{x}$; that is, $E$ does not contain any eigenvectors besides the multiples of $\vec{v}_1$ and $\vec{v}_2$; in particular, $\vec{v}_3$ is not contained in $E$. 
Fact 7.3.5 Considers the eigenvectors $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_m$ of an $n \times n$ matrix $A$, with distinct eigenvalues $\lambda_1$, $\lambda_2$, $\ldots$, $\lambda_m$. Then the $\vec{v}_i$ are linearly independent.

Proof
We argue by induction on $m$. Assume the claim holds for $m - 1$. Consider a relation

$$c_1 \vec{v}_1 + \cdots + c_{m-1} \vec{v}_{m-1} + c_m \vec{v}_m = 0$$

• apply the transformation $A$ to both sides:

$$c_1 \lambda_1 \vec{v}_1 + \cdots + c_{m-1} \lambda_{m-1} \vec{v}_{m-1} + c_m \lambda_m \vec{v}_m = 0$$

• multiply both sides by $\lambda_m$:

$$c_1 \lambda_m \vec{v}_1 + \cdots + c_{m-1} \lambda_m \vec{v}_{m-1} + c_m \lambda_m \vec{v}_m = 0$$

Subtract the above two equations:

$$c_1 (\lambda_1 - \lambda_m) \vec{v}_1 + \cdots + c_{m-1} (\lambda_{m-1} - \lambda_m) \vec{v}_{m-1} = 0$$

Since $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_{m-1}$ are linearly independent by induction, $c_i (\lambda_i - \lambda_m) = 0$, for $i = 1, \ldots, m-1$. The eigenvalues are assumed to be distinct; therefore $\lambda_i - \lambda_m \neq 0$, and $c_i = 0$. The first equation tells us that $c_m \vec{v}_m = 0$, so that $c_m = 0$ as well.
Fact 7.3.6 If an \( n \times n \) matrix \( A \) has \( n \) distinct eigenvalues, then there is an eigenbasis for \( A \). We can construct an eigenbasis by choosing an eigenvector for each eigenvalue.

EXAMPLE 6 Is there an eigenbasis for the following matrix?

\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
0 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & 3 & 4 & 5 & 6 \\
0 & 0 & 0 & 4 & 5 & 6 \\
0 & 0 & 0 & 0 & 5 & 6 \\
0 & 0 & 0 & 0 & 0 & 6
\end{bmatrix}
\]

Fact 7.3.7 Consider an \( n \times n \) matrix \( A \). If the geometric multiplicities of the eigenvalues of \( A \) add up to \( n \), then there is an eigenbasis for \( A \): We can construct an eigenbasis by choosing a basis of each eigenspace and combining these vectors.
Proof
Suppose the eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_m$, with $\dim(E_{\lambda_i}) = d_i$. We first choose a basis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{d_1}$ of $E_{\lambda_1}$, and then a basis $\vec{v}_{d_1+1}, \ldots, \vec{v}_{d_1+d_2}$ of $E_{\lambda_2}$, and so on.

Consider a relation
\[
\begin{align*}
&\underbrace{c_1 \vec{v}_1 + \cdots + c_{d_1} \vec{v}_{d_1}} + 
\underbrace{\ldots + c_{d_1+d_2} \vec{v}_{d_1+d_2}} + 
\underbrace{\ldots + \cdots + c_n \vec{v}_n} = \vec{0} \\
&\text{ where } \vec{w}_1 \text{ in } E_{\lambda_1}, \quad \vec{w}_2 \text{ in } E_{\lambda_2}, \quad \vec{w}_m \text{ in } E_{\lambda_m}
\end{align*}
\]
Each under-braced sum $\vec{w}_i$ must be a zero vector since if they are nonzero eigenvectors, they must be linearly independent and the relation can not hold.

Because $\vec{w}_1 = 0$, it follows that $c_1 = c_2 = \cdots = c_{d_1} = 0$, since $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{d_1}$ are linearly independent. Likewise, all the other $c_j$ are zero.
EXAMPLE 7 Consider an Albanian mountain farmer who raises goats. This particular breed of goats has a life span of three years. At the end of each year $t$, the farmer conducts a census of his goats. He counts the number of young goats $j(t)$ (those born in the year $t$), the middle-aged ones $m(t)$ (born the year before), and the old ones $a(t)$ (born in the year $t - 2$). The state of the herd can be represented by the vector

$$\vec{x}(t) = \begin{bmatrix} j(t) \\ m(t) \\ a(t) \end{bmatrix}$$

How do we expect the population to change from year to year? Suppose that for this breed and environment the evolution of the system can be modelled by

$$\vec{x}(t + 1) = A\vec{x}(t)$$

where

$$A = \begin{bmatrix} 0 & 0.95 & 0.6 \\ 0.8 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$$
We leave it as an exercise to interpret the entries of $A$ in terms of reproduction rates and survival rates.

Suppose the initial populations are $j_0 = 750$ and $m_0 = a_0 = 200$. What will the populations be after $t$ years, according to this model? What will happen in the long term?

**Solution**

**Step 1:** Find eigenvalues.

**Step 2:** Find eigenvectors.

**Step 3:** Express the initial vector $\vec{v}_0 = \begin{bmatrix} 750 \\ 200 \\ 200 \end{bmatrix}$ as a linear combination of eigenvectors.

**Step 4:** Write the closed formula for $\vec{v}(t)$. 
Fact 7.3.8

The eigenvalues of similar matrices Suppose matrix $A$ is similar to $B$. Then

1. Matrices $A$ and $B$ have the same characteristic polynomial; that is, $f_A(\lambda) = f_B(\lambda)$

2. $\text{rank}(A) = \text{rank}(B)$ and $\text{nullity}(A) = \text{nullity}(B)$

3. Matrices $A$ and $B$ have the same eigenvalues, with the same algebraic and geometric multiplicities. (However, the eigenvectors need not be the same.)

4. $\text{det}(A) = \text{det}(B)$ and $\text{tr}(A) = \text{tr}(B)$
Proof

a. If $B = S^{-1}AS$, then

$$f_B(\lambda) = \det(\lambda I_n - B) = \det(\lambda I_n - S^{-1}AS)$$

$$= \det(S^{-1}(\lambda I_n - A)S) = \det(S^{-1})\det(\lambda I_n - A)\det(S)$$

$$= \det(\lambda I_n - A) = f_A(\lambda)$$

b. See Section 3.4, exercise 45 and 46.

c. If follows from part (a) that matrices $A$ and $B$ have the same eigenvalues, with the same algebraic multiplicities. As for the geometric multiplicities, note that $\lambda I_n - A$ is similar to $\lambda I_n - B$ for all $\lambda$, so that $\text{nullity}(\lambda I_n - A) = \text{nullity}(\lambda I_n - B)$ by part (b).

d. These equations follow from part (a) and Fact 7.2.5. Trance and determinant are coefficients of the characteristic polynomial.