Applied Linear Algebra

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Chapter 4
Linear Spaces

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4.1 Introduction to Linear Systems

EXAMPLE 1

Consider the differential equation(DE)

$$f''(x) + f(x) = 0, or f''(x) = -f(x)$$

We are asked to find all functions $f(x)$ whose second derivative is the negative of the function itself. Recalling rules from your introductory calculus class, you will (hopefully) note that

sin(x) and cos(x)

are solutions of this DE.

Can you find any other solutions?

Definition 4.1.1

Linear spaces A linear space V is a set endowed with
(1) a rule for addition (if f and g are in V, then so is $f+g$) and
(2) a rule for scalar multiplication (if f is in V and k in R, then kf is in V)
such that these operations satisfy the following eight rules (for all f, g, h in V and all c, k in R):

1. $(f+g)+h = f+(g+h)$

2. $f+g = g+f$

3. There is a *neutral element* n in V such that $f+n=f$, for all f in V. This n is unique and denoted by 0.

4. For each f in V there is a g in V such that $f + g = 0$. this g is unique and denoted by $(-f)$

5. $k(f + g) = kf + kg$

6. $(c + k)f = cf + kf$

7. $c(kf) = (ck)f$

8. $1f = f$

EXAMPLE 2

In R^n, the prototype linear space, the neutral element is the zero vector, $\overrightarrow{0}$.

EXAMPLE 3

Let F(R,R) be the set of all functions from R to R (see Example 1), with the operations

$$(f+g)(x) = f(x) + g(x)$$

and

$$(kf)(x) = kf(x)$$

Then, F(R,R) is a linear space. The neutral element is the zero function, $f(x) = 0$ for all x.

EXAMPLE 4

If addition and scalar multiplication are given as in Definition 1.3.9, then $R^{m\times n}$, the set of all $m \times n$ matrices, is a linear space. The neutral element is the zero matrix whose entries are all zero.

EXAMPLE 5

The set of all infinite sequence of real numbers is a linear space, where addition and scalar multiplication are defined term by term:

$(x_0, x_1, x_2, \ldots) + (y_0, y_1, y_2, \ldots)$
$= (x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots)$

$$k(x_0, x_1, x_2, \ldots) = (kx_0, kx_1, kx_2, \ldots).$$

The neutral element is the sequence

$$(0, 0, 0, \ldots)$$

EXAMPLE 6

The linear equation in three unknowns,

$$ax + by + cz = d,$$

where a, b, c, and d are constants, from a linear space.

The neutral element is the equation $0 = 0$ (with $a = b = c = d = 0$).

Linear Combination

We say that an element f of a linear space is a *linear combination* of the elements $f_1, f_2, \ldots, f_n$ if

$$f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

for some scalars $c_1, c_2, \cdots, c_n$.

EXAMPLE 9

Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$. Show that $A^2 = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix}$ is a linear combination of A and I_2.

Solution

We have to find scalars c_1 and c_2 such that

$$A^2 = c_1 A + c_2 I_2,$$

or

$$A^2 = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix} = c_1 \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition 4.1.2 Subspaces

A subspace W of a linear space V is called a *subspace* of V if

1. W contains the neutral element 0 of V

2. W is closed under addition (if f and g are in W, then so is $f + g$).

3. W is closed under scalar multiplication (if f is in W and k is a scalar, then kf is in W).

We can summarize parts (2) and (3) by saying that W is closed under linear combinations.

EXAMPLE 10
Show that the polynomials of degree$\leq$ 2, of the form $f(x) = a + bx + cx^2$, are a subspace W of the space F(R,R) of all functions from R to R.

EXAMPLE 11
Show that the differentiable functions form a subspace W of F(R,R)

EXAMPLE 12
Here are three more subspaces of F(R,R):

1. C^∞, the smooth functions, that is, functions we can differentiate as many times as we want. This subspace contains all polynomials, exponential functions, $sin(x)$, and $cos(x)$, for example.

2. P, the set of all polynomials.

3. P_n, the set of all polynomials of degree $\leq n$

EXAMPLE 13

Show that the matrices B that commute with $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ form a subspace of $R^{2\times 2}$.

Solution

(a) The zero matrix 0 commutes with A.

(b) If matrices B_1 and B_2 commute with A, then so does matrix $B_1 + B_2$.

(c) If B commutes with A, then so does kB.

EXAMPLE 14

Consider the set W of all noninvertible 2×2 matrices. Is W a subsequence of $R^{2\times 2}$?

Solution

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition 4.1.3
Span, linear independence, basis, coordinates

Consider the elements $f_1, f_2, \ldots, f_n$ of a linear space V.

1. We say that $f_1, f_2, \ldots, f_n$ *span* V if every f in V can be expressed as a linear combination of $f_1, f_2, \ldots, f_n$.

2. We say that $f_1, f_2, \ldots, f_n$ are (*linearly*) *independent* if the equation

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

has only the trivial solution

$$c_1 = c_2 = \cdots = c_n = 0.$$

3. We say that elements $f_1, f_2, \ldots, f_n$ are a *basis* of V if they span V and are independent. This means that every f in V can be written uniquely as a linear combination

$$f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n.$$

The coefficients $c_1, c_2, \ldots, c_n$ are called the *coordinates* of f with respect to the basis $f_1, f_2, \ldots, f_n$.

Fact 4.1.4 Dimension

If a linear space V has a basis with n elements, then all other bases of V consist of n elements as well. We say that n is the *dimension* of V:

$$dim(V) = n.$$

EXAMPLE 15

Find a basis of $V = R^{2\times 2}$ and thus determine $dim(V)$.

Solution

We can write any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

EXAMPLE 16

Find a basis of P_2, the space of all polynomials of degree ≤ 2, and thus determine the dimension of P_2.

Solution

We can write any polynomial $f(x)$ of degree ≤ 2 uniquely as:

$$f(x) = a + bx + cx^2 = a \cdot 1 + b \cdot x + c \cdot x^2$$

EXAMPLE 17

Find a basis of the space V of all matrices B that commute with $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$.

Solution

We need to find all matrices $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} 2b & a+3b \\ 2d & c+3d \end{bmatrix} = \begin{bmatrix} c & d \\ 2a+3c & 2b+3d \end{bmatrix}$$

$$c = 2b, d = a + 3b$$

So a typical matrix B in V is of the form

$$B = \begin{bmatrix} a & b \\ 2b & a+3b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= aI_2 + bA$$

The matrices I_2 and A form a basis of V, so that dim(V)=2.

EXAMPLE 19

Let $f_1, f_2, \ldots, f_n$ be polynomials. Explain why these polynomials do not span the space P of all polynomials.

Solution

Let N be the maximum of the degrees of these n polynomials. Then all linear combinations of $f_1, f_2, \ldots, f_n$ are in P_N, the space of the polynomials of degree $\leq N$. Any polynomial of higher degree, such as $f(x) = x^{N+1}$, will not be in the span of $f_1, f_2, \ldots, f_n$.

This implies that the space P of all polynomials does not have a finite basis $f_1, f_2, \ldots, f_n$.

Definition 4.1.6 Finite-dimensional linear spaces

A linear spaces V is called $finite-dimensional$ if it has a (finite) basis $f_1, f_2, \ldots, f_n$, so that we can define its dimension $dim(V) = n$. (See Definition 4.1.4.) Otherwise, the space is called $infinite-dimensional$.

Exercises 4.1: 3, 5, 7, 8, 17, 18, 20, 33, 35

*__EXAMPLE 7__

Consider the plane with a point designated as the origin, O, but without a coordinate system (the coordinate-free plane).

- A *geometric vector* $\overrightarrow{v}$ in this plane is an arrow (a directed line segment) with its tail at the origin, as shown in Figure 1.

- The sum $\overrightarrow{v} + \overrightarrow{w}$ of vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ is defined by means of a parallelogram, as illustration in Figure 2.

- If k is a positive scalar, then vector $k\overrightarrow{v}$ points in the same direction as $\overrightarrow{v}$, but $k\overrightarrow{v}$ is k times as long as $\overrightarrow{v}$; see Figure 3.

- If k is negative, then $k \overrightarrow{v}$ points in the opposite direction, and it is $\mid k \mid$ times as long as $\overrightarrow{v}$; see Figure 4.

The geometric vectors in the plane with these operations forms a linear space.

The neutral element is the zero vector $\overrightarrow{0}$, with tail and head at the origin.

By introducing a *coordinate system*, we can identify the plane of geometric vectors with R^2; this was the great idea of Descartes' *Analytic Geometry*. In Section 4.3, we will study this idea more systematically.

***EXAMPLE 8**

Let C be the set of the *complex numbers*. We trust that you have at least a fleeting acquaintance with complex numbers. Without attempting a definition, we recall that a complex number can be expressed as $z = a + bi$, where a and b are real numbers. Addition of complex numbers is defined in a natural way, by the rule

$$(a + bi) + (c + di) = (a + c) + i(b + d).$$

If k is a real scalar, we define

$$k(a + bi) = ka + i(kb).$$

There is also a (less natural) rule for the multiplication of complex numbers, but we are not concerned with this operation here.

The complex numbers C with the two operations just given form a linear space; the neutral element is the complex number $0 = 0 + 0i$.

*__Fact 4.1.5 Linear differential equations__

The solutions of the DE

$$f^{''}(x) + af^{'}(x) + bf(x) = 0$$

where a and b are constants, form a two-dimensional subspace of the space C^∞ of smooth functions.

More generally, the solutions of the DE

$$f^{(n)}(x) + a_{n-1}f^{n-1}(x) + \cdots + a_1 f^{'}(x) + a_0 f(x)$$

(where the a_i are constants) form an n-dimensional subspace of C^∞. A DE of this is called an nth-order linear differential equation.

Fact 4.1.5 will be proven in Section 9.3.

***EXAMPLE 18**

Find all solutions of the DE

$$f''(x) + f'(x) - 6f(x) = 0.$$

($Hint$: Find all exponential functions $f(x) = e^{kx}$ that solve the DE)

An exponential function $f(x) = e^{kx}$ solves the DE if $k = 2$ or $k = -3$. Since

$$k^2 e^{kx} + ke^{kx} - 6e^{kx} = (k^2 + k - 6)e^{kx}$$

$$= (k + 3)(k - 2)e^{kx} = 0$$

According to Fact 4.1.5, the solution space V is two-dimensional. Thus, the two exponential functions e^{2x} and e^{-3x} form a basis of V, and all solutions are of the form

$$f(x) = c_1 e^{2x} + c_2 e^{-3x}$$