3.4 COORDINATES

EXAMPLE 1
Let $V$ be the plane in $\mathbb{R}^3$ with equation $x_1 + 2x_2 + 3x_3 = 0$, a two-dimensional subspace of $\mathbb{R}^3$. We can describe a vector in this plane by its spatial (3D) coordinates; for example, vector

$$\vec{x} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$$

is in plane $V$. However, it may be more convenient to introduce a plane coordinate system in $V$.

Consider any two vectors in plane $V$ that aren’t parallel, e.g.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$
See Figure 1, where we label the new axes $c_1$ and $c_2$, with the new coordinate grid defined by vectors $\vec{v}_1$ and $\vec{v}_2$.

Note that the $c_1\ -\ c_2$ coordinates of vector $\vec{v}_1$ is $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the coordinates of vector $\vec{v}_2$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively.

For a vector $\vec{x}$ in plane $V$, we can find the scalars $c_1$ and $c_2$ such that

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2.$$ 

For example, $\vec{x} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$
Therefore, the $c_1 - c_2$ coordinates of $\vec{x}$ are

\[
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} = \begin{bmatrix}
  3 \\
  2
\end{bmatrix}
\]

See Figure 3.

Let’s denote the basis $\vec{v}_1$, $\vec{v}_2$ of $V$ by $B$ (Fraktur B). Then, the coordinate vector of $\vec{x}$ with respect to $B$ is denoted by $[\vec{x}]_B$:

If $\vec{x} = \begin{bmatrix}
  5 \\
  -1 \\
  -1
\end{bmatrix}$, then $[\vec{x}]_B = \begin{bmatrix}
  3 \\
  2
\end{bmatrix}$
**Definition 3.4.1**

**Coordinates in a subspace of \( \mathbb{R}^n \)**

Consider a basis \( B \) of a subspace \( V \) of \( \mathbb{R}^n \), consisting of vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \). Any vector \( \vec{x} \) in \( V \) can be written uniquely as

\[
\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_m \vec{v}_m
\]

The scalars \( c_1, c_1, \ldots, c_m \) are called the \( B \)-coordinates of \( \vec{x} \), and the vector

\[
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_m
\end{bmatrix}
\]

is called the \( B \)-coordinate vector of \( \vec{x} \), denoted by \( \begin{bmatrix} \vec{x} \end{bmatrix}_B \).

Note that

\[
\vec{x} = S \begin{bmatrix} \vec{x} \end{bmatrix}_B
\]

where \( S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_m \end{bmatrix} \), an \( n \times m \) matrix.
EXAMPLE 2
Consider the basis $B$ of $\mathbb{R}^2$ consisting of vectors
\[
\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}
\]
a. If $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, find $[\vec{x}]_B$

b. If $[\vec{x}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $\vec{x}$

Solution
a. To find the coordinates of vector $\vec{x}$, we need to write $\vec{x}$ as a linear combination of the basis vectors:

\[
\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2, \quad \text{or} \quad \begin{bmatrix} 10 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}
\]

Alternatively, we can solve the equation

\[
\vec{x} = S [\vec{x}]_B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} [\vec{x}]_B
\]
for $[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$
\[
\begin{bmatrix}
\tilde{x}_B
\end{bmatrix} = S^{-1} \tilde{x} = \begin{bmatrix}
3 & -1 \\
1 & 3
\end{bmatrix}^{-1} \begin{bmatrix}
10 \\
10
\end{bmatrix} = \frac{1}{10} \begin{bmatrix}
3 & 1 \\
-1 & 3
\end{bmatrix} \begin{bmatrix}
10 \\
10
\end{bmatrix} = \begin{bmatrix}
4 \\
2
\end{bmatrix}
\]

b. By definition of coordinates, \( \begin{bmatrix}
\tilde{x}_B
\end{bmatrix} = \begin{bmatrix}
2 \\
-1
\end{bmatrix} \) means that
\[
\tilde{x} = 2\tilde{v}_1 + (-1)\tilde{v}_2 = 2 \begin{bmatrix}
3 \\
1
\end{bmatrix} + (-1) \begin{bmatrix}
-1 \\
3
\end{bmatrix} = \begin{bmatrix}
7 \\
-1
\end{bmatrix}
\]

Alternatively, use the formula
\[
\tilde{x} = S \begin{bmatrix}
\tilde{x}_B
\end{bmatrix} = \begin{bmatrix}
3 & -1 \\
1 & 3
\end{bmatrix} \begin{bmatrix}
2 \\
-1
\end{bmatrix} = \begin{bmatrix}
7 \\
-1
\end{bmatrix}
\]
EXAMPLE 3

Let $L$ be the line in $R^2$ spanned by vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Let $T$ be the linear transformation from $R^2$ to $R^2$ that projects any vector orthogonally onto line $L$, as shown in Figure 5.

1. In $\vec{x}_1 - \vec{x}_2$ coordinate system (See Figure 5): Sec 2.2 (pp. 59).

2. In $c_1 - c_2$ coordinate system (See Figure 6):

   $T$ transforms vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ into $\begin{bmatrix} c_1 \\ 0 \end{bmatrix}$.

   That is, $T$ is given by the matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$

The transforms from $\begin{bmatrix} \vec{x} \end{bmatrix}_B$ into $\begin{bmatrix} T(\vec{x}) \end{bmatrix}_B$ is called the $B$-matrix of $T$:

$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_B = B \begin{bmatrix} \vec{x} \end{bmatrix}_B$
Definition 3.4.2
The $B$-matrix of a linear transformation
Consider a linear transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ and a basis $B$ of $\mathbb{R}^n$. The $n \times n$ matrix $B$ that transforms $\begin{bmatrix} \vec{x} \end{bmatrix}_B$ into $\begin{bmatrix} T(\vec{x}) \end{bmatrix}_B$ is called the $B$-matrix of $T$:

$$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_B = B \begin{bmatrix} \vec{x} \end{bmatrix}_B$$

for all $\vec{x}$ in $\mathbb{R}^n$.

Fact 3.4.3 The columns of the $B$-matrix of a linear transformation
Consider a linear transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ and a basis $B$ of $\mathbb{R}^n$ consisting of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Then, the $B$-matrix of $T$ is

$$B = \begin{bmatrix} \begin{bmatrix} T(\vec{x}_1) \end{bmatrix}_B & \begin{bmatrix} T(\vec{x}_2) \end{bmatrix}_B & \cdots & \begin{bmatrix} T(\vec{x}_n) \end{bmatrix}_B \end{bmatrix}$$

That is, the columns of $B$ are the $B$-coordinate vectors of $T(\vec{v}_1)$, $T(\vec{v}_2)$, $\ldots$, $T(\vec{v}_n)$. 
EXAMPLE 4
Consider two perpendicular unit vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^3$. Form the basis $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$ of $\mathbb{R}^3$; let’s denote this basis by $B$. Find the B-matrix $B$ of the linear transformation $T(\vec{x}) = \vec{v}_1 \times \vec{x}$.

(see Exercise 2.1: 44 on pp. 49,
\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix} \times \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix} = \begin{bmatrix}
a_2b_3 - a_3b_2 \\
a_3b_1 - a_1b_3 \\
a_1b_2 - a_2b_1
\end{bmatrix}
\]

Solution
Use Fact 3.4.3 to construct $B$ column by column:

\[
B = \begin{bmatrix}
T(\vec{x}_1) \\
T(\vec{x}_2) \\
\vdots \\
T(\vec{x}_n)
\end{bmatrix}_B = \begin{bmatrix}
\vec{v}_1 \times \vec{v}_1 \\
\vec{v}_1 \times \vec{v}_2 \\
\vec{v}_1 \times \vec{v}_3 \\
\vec{0}
\end{bmatrix}_B = \begin{bmatrix}
0 \\
0 \\
-1 \\
0
\end{bmatrix}_B
= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}
\]
EXAMPLE 5

Let $T$ be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that projects any vector orthogonally onto the line $L$ spanned by $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. In Example 3, we found that the matrix of $T$ with respect to the basis $B$ consisting of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

What is the relationship between $B$ and the standard matrix $A$ of $T$ (such that $T(\vec{x}) = A\vec{x}$)?

Solution

Recall from Definition 3.4.1 that

$$\vec{x} = S \begin{bmatrix} \vec{x} \end{bmatrix}_B, \text{ where } S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

and consider the following diagram: (Figure 7)
Note that $T(\vec{x}) = AS \begin{bmatrix} \vec{x} \end{bmatrix}_B$ and also $T(\vec{x}) = SB \begin{bmatrix} \vec{x} \end{bmatrix}_B$, so that $AS \begin{bmatrix} \vec{x} \end{bmatrix}_B = SB \begin{bmatrix} \vec{x} \end{bmatrix}_B$ for all $\vec{x}$.

Thus,

$$AS = SB \text{ and } A = SB S^{-1}$$

Now we can find the standard matrix $A$ of $T$:

$$A = SB S^{-1}
= \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/10 & 3 \\ -1 & 3 \end{bmatrix}
= \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}$$

Alternatively, we could use Fact 2.2.5 to construct matrix $A$. The point here was to explore the relationship between matrices $A$ and $B$. 
Fact 3.4.4
Standard matrix versus $B$-matrix of a linear transformation
Consider a linear transformation $T$ from $R^n$ to $R^n$ and a basis $B$ of $R^n$ consisting of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Let $B$ be the $B$-matrix of $T$ and let $A$ be the standard matrix of $T$ (such that $T(\vec{x})=A\vec{x}$). Then, $AS=SB$, $B=S^{-1}AS$, and $A=SBS^{-1}$, where

$$S = \begin{bmatrix} \vert & \vert & \vert \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_m \\ \vert & \vert & \vert \end{bmatrix}$$

Definition 3.4.5 Similar matrices
Consider two $n \times n$ matrices $A$ and $B$. We say that $A$ is similar to $B$ if there is an invertible matrix $S$ such that

$$AS=SB, \text{ or } B=S^{-1}AS$$
EXAMPLE 6
Is matrix \( A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \) similar to \( B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \) ?

Solution
We are looking for a matrix \( S = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \) such that \( AS = SB \), or
\[
\begin{bmatrix} x + 2z & y + 2t \\ 4x + 3z & 4y + 3t \end{bmatrix} = \begin{bmatrix} 5x & -y \\ 5z & -t \end{bmatrix}.
\]
These equations simplify to
\[
z = 2x, \quad t = -y,
\]
so that any invertible matrix of the form
\[
S = \begin{bmatrix} x & y \\ 2x & -y \end{bmatrix}
\]
does the job. Note that \( \det(S) = -3xy \). Matrix \( S \) is invertible if \( \det(S) \neq 0 \) (i.e., if neither \( x \) nor \( y \) is zero).
EXAMPLE 7
Show that if matrix $A$ is similar to $B$, then its power $A^t$ is similar to $B^t$ for all positive integers $t$. (That is, $A^2$ is similar to $B^2$, $A^3$ is similar to $B^3$, etc.)

Solution
We know that $B = S^{-1}AS$ for some invertible matrix $S$. Now, $B^t$

$$= \underbrace{(S^{-1}AS)(S^{-1}AS)\ldots(S^{-1}AS)(S^{-1}AS)}_{t-times}$$

$$= S^{-1}A^tS,$$

proving our claims. Note the cancellation of many terms of the form $SS^{-1}$. 
Fact 3.4.6
Similarity is an equivalence relation

1. An $n \times n$ matrix $A$ is similar to itself (Reflexivity).

2. If $A$ is similar to $B$, then $B$ is similar to $A$ (Symmetry).

3. If $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$ (Transitivity).

Proof
$A$ is similar to $B$: $B = P^{-1}AP$
$B$ is similar to $C$: $C = Q^{-1}BQ$, then

$C = Q^{-1}BQ = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ)$

that is, $A$ is similar to $C$ by matrix $PQ$.

Homework Exercise 3.4: 5, 6, 9, 10, 13, 14, 19, 31, 39