3.3 The Dimension of a Subspace of R^n

Fact 3.3.2
All bases of a subspace V of R^n consist of the same number of vectors.

Hint Basis: linear independent and span V (Def 3.2.3)

Fact 3.3.1
Consider vectors $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_p}$ and $\vec{w_1}$, $\vec{w_2}$, ..., $\vec{w_q}$ in a subspace V of R^n. If the vectors $\vec{v_i}$ are linearly independent, and the vectors $\vec{w_j}$ span V, then p $\leq$ q.

Proof 3.3.2

Consider two bases $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_p}$ and $\vec{w_1}$, $\vec{w_2}$, ..., $\vec{w_q}$ of V. Since the $\vec{v_i}$ are linearly independent, and the vectors $\vec{w_j}$ span V, we have $p \leq q$. Like wise, since the $\vec{w_j}$ are linearly independent and the $\vec{v_i}$ span V, we have $q \leq p$. Therefore, $p = q$.

Proof 3.3.1

$$\begin{array}{ccccc} \vec{v}_1 & = & a_{11}\vec{w}_1 & + \cdots + & a_{1q}\vec{w}_q \\ \vdots & & \vdots & & \vdots \\ \vec{v}_p & = & a_{p1}\vec{w}_1 & + \cdots + & a_{pq}\vec{w}_q \end{array}$$

Write each of these equations in matrix form:

$$\begin{bmatrix} | & & | \\ \vec{w}_1 & \dots & \vec{w}_q \\ | & & | \end{bmatrix} \begin{bmatrix} a_{11} \\ \vdots \\ a_{1q} \end{bmatrix} = \vec{v}_1$$

$$\dots$$

$$\begin{bmatrix} | & & | \\ \vec{w}_1 & \dots & \vec{w}_q \\ | & & | \end{bmatrix} \begin{bmatrix} a_{p1} \\ \vdots \\ a_{pq} \end{bmatrix} = \vec{v}_p$$

Combine all these equations into one matrix equation:

$$\begin{bmatrix} | & & | \\ \vec{w}_1 & \dots & \vec{w}_q \\ | & & | \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{p1} \\ \vdots & & \vdots \\ a_{1q} & \dots & a_{pq} \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_p \\ | & & | \end{bmatrix}$$

$$MA = N$$

Because

$$A\vec{x} = \vec{0}, MA\vec{x} = N\vec{x} = \vec{0}$$

The kernel of A is contained in the kernel of N.

Since the kernel of N is $\{\vec{0}\}$ (since the $\vec{v}_i$ are linearly independent), the kernel of A is $\{\vec{0}\}$ as well.

This implies that $rank(A) = p \leq q$ (by Fact 3.1.7).

Definition. Dimension
Consider a subspace V of R^n. The number of vectors in a basis of V is called the **dimension** *of V, denoted by $dim(V)$.*

What is the dimension R^n itself?

Clearly, R^n ought to have dimension n. This is indeed the case: the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ form a basis of R^n called its **standard basis**.

A plane E in R^3 is two-dimensional.

Fact 3.3.4

Consider a subspace V of R^n with $dim(V) = m$

1. We can find at most m linearly independent vectors in V.

2. We need at least m vectors to span V.

3. If m vectors in V are linearly independent, then they form a basis of V.

4. If m vectors span V, then they form a basis of V.

Proof 3.3.4 (3)
Consider linearly independent vectors $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_m}$ in V. We have to show that the $\vec{v}_i$ span V. Pick a $\vec{v}$ in V. Then the vectors $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_m}, \vec{v}$ will be linearly dependent, by (1). Therefore, there is a nontrivial relation

$$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m + c\vec{v} = \vec{0}$$

We can solve the relation for $\vec{v}$ and express it as a linear combination of the $\vec{v}_i$. In other words, the $\vec{v}_i$ span V.

Finding a Basis of the Kernel

Example. *Find a basis of the kernel of the following matrix, and determine the dimension of the kernel:*

$$A = \left[\begin{array}{ccccc} 1 & 2 & 0 & 3 & 0 \\ 2 & 4 & 1 & 9 & 5 \end{array} \right]$$

Solution

$$A = \left[\begin{array}{ccccc} 1 & 2 & 0 & 3 & 0 \\ 2 & 4 & 1 & 9 & 5 \end{array} \right] \begin{array}{c} \\ -2(I) \end{array}$$

$$\longrightarrow rref(A) = \left[\begin{array}{ccccc} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 & 5 \end{array} \right]$$

This corresponds to the system

$$\left| \begin{array}{lllll} x_1 + 2x_2 & & 3x_4 & & = 0 \\ & x_3 + & 3x_4 & +5x_5 & = 0 \end{array} \right|$$

with general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s-3t \\ s \\ -3t-5r \\ t \\ r \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad \uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

$$\qquad\qquad\qquad\qquad \vec{v}_1 \qquad\qquad \vec{v}_2 \qquad\qquad \vec{v}_3$$

The tree vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ span $ker(A)$ and form a basis of the kernel of A (i.e. linearly independent).

dim(ker A)=(number of nonleading variables)
=(number of columns of A)-(number of leading variables)
=(number of columns of A)-rank(A)
=5-2 =3

Fact 3.3.5
Consider an $m \times n$ matrix A.

$$dim(kerA) = n - rank(A)$$

Finding a Basis of the Image

Example. *Find a basis of the image of the linear transformation T from R^5 to R^4 with matrix*

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}$$

and determine the dimenson of the image.

Solution

We know the columns of A span the image of A, but they are linearly dependent in this example. To construct a basis of im(A), we could find a relation among the columns of A, express one of the columns as linear combinartion of the others, and then omit this vector as redundant.

We first find the reduced row-echelon form of A:

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{matrix}$$

$$E = rref(A) = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 \end{matrix}$$

By inspection, we can express any column of $rref(A)$ that does not contain a leading 1 as a linear combination of earlier columns that do contain a leading 1.

$$\vec{w}_3 = \vec{w}_1 - 2\vec{w}_2, \text{ and } \vec{w}_4 = 2\vec{w}_1 - 3\vec{w}_2$$

It may surprise you that the same relationships hold among the corresponding columns of the matrix A.

$$\vec{v}_3 = \vec{v}_1 - 2\vec{v}_2, \text{ and } \vec{v}_4 = 2\vec{v}_1 - 3\vec{v}_2$$

Since $\vec{w_1}$, $\vec{w_2}$, and $\vec{w_5}$ are linearly independent, so are the vectors $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_5}$. (Why?)

The vectors $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_5}$ alone span the image of A, since any vector $\vec{v}$ in the image of A can be expressed as

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4} + c_5\vec{v_5}$$

$$= c_1\vec{v_1} + c_2\vec{v_2} + c_3(\vec{v_1} - 2\vec{v_2}) + c_4(2\vec{v_1} - 3\vec{v_2}) + c_5\vec{v_5}$$

Therefore, the vectors $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_5}$ form a basis of $im(A)$, and thus $dim(imA) = 3$.

Definition.

A column of a matrix A is called a **pivot column** if the corresponding column of $rref(A)$ contains a leading 1.

Fact 3.3.7 The pivot columns of a matrix A form a basis of $im(A)$.

Fact 3.3.8 For any matrix A,

$$rank(A) = dim(imA).$$

Fact 3.3.9 Rank-Nullity Theorem
If A is an $m \times n$ matrix, then

$$dim(ker A) + dim(im A) = n.$$

The dimension of the kernel of matrix A is called the **nullity** of A:

$$nullity(A) = dim(ker A).$$

Using this definition and Fact 3.3.8, we can write:

$$nullity(A) + rank(A) = n.$$

$\Rightarrow$ The larger the kernel, the smaller the image, and vice versa.

Bases of R^n

How can we tell n given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in R^n form a basis?

The $\vec{v}_i$ form a basis of R^n if every vector $\vec{b}$ in R^n can be written uniquely as a linear combination of the $\vec{v}_i$:

$$\vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The linear system

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{b}$$

has a unique solution if (only if) the $n \times n$ matrix

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

is invertible.

Fact 3.3.10 The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in R^n form a basis of R^n if (and only if) the matrix

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

is invertible.

Example. *Are the following vectors a basis of R^4?*

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 9 \\ 1 \end{bmatrix}, \ \vec{v_2} = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 8 \end{bmatrix}, \ \vec{v_3} = \begin{bmatrix} 1 \\ 8 \\ 1 \\ 5 \end{bmatrix}, \ \vec{v_4} = \begin{bmatrix} 1 \\ 9 \\ 7 \\ 3 \end{bmatrix}$$

Solution

We have to check whether the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 9 \\ 9 & 4 & 1 & 7 \\ 1 & 8 & 5 & 3 \end{bmatrix}$$

is invertible. Using technology, we find that

$$\text{reff} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 9 \\ 9 & 4 & 1 & 7 \\ 1 & 8 & 5 & 3 \end{bmatrix} = I_4$$

Thus, the vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$, $\vec{v_4}$ form a basis of R^4

Summary 3.3.11

Consider an $n \times n$ matrix

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

Then the following statements are equivalent:

1. A is invertible.

2. The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, for all $\vec{b}$ for all $\vec{b}$ in R^n.

3. $rref(A) = I_n$.

4. $rank(A) = n$.

5. $im(A) = R^n$.

6. $ker(A) = \{\vec{0}\}$.

7. The $\vec{v}_i$ are a basis of R^n.

8. The $\vec{v}_i$ span R^n.

9. The $\vec{v}_i$ are linearly independent.

Homework 3.3 *6, 7, 8, 17, 18, 27, 31, 33, 39, 58, 59*

Exercise 49: Find a basis of the row space of the matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 51: Consider an arbitrary $m \times n$ matrix A.

1. What is the relationship between the row spaces of A and $E = rref(A)$?

2. What is the relationship between the dimension of the row space of A and the rank of A?