Applied Linear Algebra

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Chapter 6
Determinants

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6.1 INTRODUCTION TO DETERMINANTS

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff

$$ad - bc \neq 0,$$

The quantity $ad - bc$ is called the determinant of the matrix A.

Can we assign a number $det(A)$ to any square matrix A, such that A is invertible iff $det(A) \neq 0$?

The determinants of a 3×3 matrix
Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The matrix is **not** invertible if the three column vectors are contained in a same plane.

In this case, one of the vector $\vec{u}$ is perpendicular to the cross product $\vec{v} \times \vec{w}$; that is,

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \cdot \left(\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \times \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \right)$$

$$= \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \cdot \begin{bmatrix} a_{22}a_{33} - a_{32}a_{23} \\ a_{32}a_{13} - a_{12}a_{33} \\ a_{12}a_{23} - a_{22}a_{13} \end{bmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23})$$
$$+ a_{21}(a_{32}a_{13} - a_{12}a_{33})$$
$$+ a_{31}(a_{12}a_{23} - a_{22}a_{13})$$

The terms $(a_{22}a_{33} - a_{32}a_{23}), (a_{32}a_{13} - a_{12}a_{33})$, and $(a_{12}a_{23} - a_{22}a_{13})$ are the determinants of submatrices of A.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Definition 6.2.9 Minors
For an $n \times n$ matrix A, let A_{ij} be the matrix obtained by omitting the ith row and the jth column of A. The $(n-1) \times (n-1)$ matrix A_{ij} is called a minor of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

We can now represent the determinant of a 3×3 matrix more succinctly:

$$det(A) = a_{11}det(A_{11}) - a_{21}det(A_{21}) + a_{31}det(A_{31})$$

This representation of the determinant is called the **Laplace expansion** of det(A) *down the first column*. Like wise, we can expand along the first row:

$$det(A) = a_{11}det(A_{11}) - a_{12}det(A_{12}) + a_{13}det(A_{13})$$

In fact, we can expand along any row or down any column.

The rule for the signs is as follows: The summand $a_{ij}det(A_{ij})$ has a negative sign if the sum of the two indices, $i + j$, is odd.

Fact 6.2.10 Laplace expansion
We can compute the determinant of an $n \times n$ matrix A by Laplace expansion along any row or down any column.
Expansion along the ith row:

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} det(A_{ij})$$

Expansion down the jth column:

$$det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} det(A_{ij})$$

Example Use Laplace expansion to compute $det(A)$ for

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 9 & 1 & 3 & 0 \\ 9 & 2 & 2 & 0 \\ 5 & 0 & 0 & 3 \end{bmatrix}$$

Solution Looking for rows or columns with as many zeros as possible, we can choose the second column:

$$det(A) = 1 det A_{22} - 2 det A_{32}$$

$$= 1 det \begin{bmatrix} 1 & 0 & 1 & 2 \\ 9 & 1 & 3 & 0 \\ 9 & 2 & 2 & 0 \\ 5 & 0 & 0 & 3 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 1 & 2 \\ 9 & 1 & 3 & 0 \\ 9 & 2 & 2 & 0 \\ 5 & 0 & 0 & 3 \end{bmatrix}$$

$$= det \begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix} - 2 det \begin{bmatrix} 1 & 1 & 2 \\ 9 & 3 & 0 \\ 5 & 0 & 3 \end{bmatrix}$$

$$= 2 det \begin{bmatrix} 9 & 2 \\ 5 & 0 \end{bmatrix} + 3 det \begin{bmatrix} 1 & 1 \\ 9 & 2 \end{bmatrix} - 2 \left(2 det \begin{bmatrix} 9 & 3 \\ 5 & 0 \end{bmatrix} + 3 det \begin{bmatrix} 1 & 1 \\ 9 & 3 \end{bmatrix} \right)$$

$$= -20 - 21 - 2(-30 - 18) = 55$$

6.2 PROPERTIES OF THE DETERMINANT

Fact 6.2.1 Determinant of the transpose
If A is a square matrix, then

$$det(A^T) = det(A).$$

Linearity Properties of the Determinant
The function $T(A) = det(A)$ from $R^{n\times n}$ to R is nonlinear (if $n > 1$). Still, the determinant has some noteworthy linearity properties.

EXAMPLE 1 Consider the transformation

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = det\begin{bmatrix} 1 & 2 & x_1 & 3 \\ 4 & 5 & x_2 & 6 \\ 7 & 6 & x_3 & 5 \\ 4 & 3 & x_4 & 1 \end{bmatrix}$$

from R^4 to R. Is this transformation linear?

Solution Since

$$det \begin{bmatrix} 1 & 2 & x_1 & 3 \\ 4 & 5 & x_2 & 6 \\ 7 & 6 & x_3 & 5 \\ 4 & 3 & x_4 & 1 \end{bmatrix} = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$$

for some constants c_i, the transformation T is linear.

Fact 6.2.3 Linearity of the determinant

(a) If three $n \times n$ matrix A, B, C are the same, except for the jth column and the jth column of C is the jth columns of A and B, then $det(C) = det(A) + det(B)$:

$$det \underbrace{\begin{bmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{x}+\vec{y} & \cdots & \vec{v}_n \\ | & & | & & | \end{bmatrix}}_{A}$$

$$= det \underbrace{\begin{bmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{x} & \cdots & \vec{v}_n \\ | & & | & & | \end{bmatrix}}_{B} + det \underbrace{\begin{bmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{y} & \cdots & \vec{v}_n \\ | & & | & & | \end{bmatrix}}_{C}$$

(b) If two $n \times n$ matrix A, B are the same, except for the jth column and the jth column of B is k times the jth columns of A, then $det(B) = kdet(A)$:

$$det \underbrace{\begin{bmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & k\vec{x} & \cdots & \vec{v}_n \\ | & & | & & | \end{bmatrix}}_{B} = kdet \underbrace{\begin{bmatrix} | & & | & & | \\ \vec{v}_1 & \cdots & \vec{x} & \cdots & \vec{v}_n \\ | & & | & & | \end{bmatrix}}_{A}$$

Determinants and Gauss-Jordan Elimination

There are three elementary row operations:
(a) dividing a row by a scalar,
(b) swapping two rows, and
(c) adding a multiple of a row to another row.

(a) If

$$A = \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_i & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix} \text{ and } B = \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_i/k & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix}$$

then det(B)=(1/k)det(A), by linearity in the ith row.

(b) If the matrix B is obtained from A by swapping any two rows, then det(B)=-det(A).

(c) $A = \begin{bmatrix} & \vdots & \\ - & \vec{v}_i & - \\ & \vdots & \\ - & \vec{v}_j & - \\ & \vdots & \end{bmatrix} \longrightarrow B = \begin{bmatrix} & \vdots & \\ - & \vec{v}_i & - \\ & \vdots & \\ - & \vec{v}_j + k\vec{v}_i & - \\ & \vdots & \end{bmatrix}$

By linearity in the jth row, we find that

$$det(B) = det \begin{bmatrix} & \vdots & \\ - & \vec{v}_i & - \\ & \vdots & \\ - & \vec{v}_j & - \\ & \vdots & \end{bmatrix} + kdet \begin{bmatrix} & \vdots & \\ - & \vec{v}_i & - \\ & \vdots & \\ - & \vec{v}_i & - \\ & \vdots & \end{bmatrix}$$

$= det(A) + 0 = det(A)$

Proof

If a matrix A has two equal rows, what can you say about det(A)? Since we have swapped two equal rows, we have $B = A$

$$det(A) = det(B) = -det(A),$$

so that det(A)=0.

Fact 6.2.4 Elementary row operations and determinants

a. If B is obtained from A by dividing a row of A by a scalar k, then

$$det(B) = (1/k)det(A).$$

b. If B is obtained from A by a row swap, then

$$det(B) = -det(A).$$

c. If B is obtained from A by adding a multiple of a row of A to another row, then

$$det(B) = det(A).$$

Analogous results hold for elementary column operations.

Suppose that in the course of Gauss-Jordan elimination, we swap rows s times and divide various rows by the scalars $k_1, k_2, ..., k_r$. Then

$$det(rref A) = (-1)^s \frac{1}{k_1 k_2 ... k_r} det(A),$$

or

$$det(A) = (-1)^s k_1 k_2 ... k_r det(rref A).$$

(a) When A is invertible, then $rref(A) = I_n$, so that det(rref A)=1, and

$$det(A) = (-1)^s k_1 k_2 ... k_r. \neq 0$$

(b) When A is not invertible, then det(A)=0.

Algorithm 6.2.6
Consider an invertible matrix A. Suppose you swap rows s times and you divide various rows by the scalars $k_1, k_2, ..., k_r$ as you row-reduce A. Then,

$$det(A) = (-1)^s k_1 k_2 ... k_r$$

The Determinant of a Product

Fact 6.1.3
The determinants of an (upper or lower) triangular matrix is the product of the diagonal entries of the matrix.

Fact 6.2.1 Determinant of a transpose
If A is a square matrix, then

$$det(A^T) = det(A)$$

Fact 6.2.7 Determinant of a product
If A and B are $n \times n$ matrices, then

$$det(AB) = det(A)det(B)$$

The Determinant of an Inverse

Fact 6.2.8 Determinant of an inverse
If A is an invertible matrix, then

$$det(A^{-1}) = (detA)^{-1} = \frac{1}{det(A)}.$$

Preliminary for Proof of Fact 6.2.4

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Compute E_1A, E_2A, and E_3A,

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix},$$

$$E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}, E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}$$

we found that addition of -4 times row 1 of A to row 3 produces E_1A. (This is a row replacement operation.) An interchange of rows 1 and 2 of A produces E_2A, and multiplication of row 3 of A by 5 produces E_3A.

Proof of Fact 6.2.4

If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$detEA = (detE)(detA)$$

where

$$detE = \begin{cases} k & if\ E\ is\ a\ scale\ by\ k \\ -1 & if\ E\ is\ an\ interchange \\ 1 & if\ E\ is\ a\ row\ replacement \end{cases}$$

The proof is by induction on the size of A. The case of a 2×2 matrix can be verified. Suppose that the theorem hold for determinants of $n\times n$ matrices with $k \geq 2$, let A be $(n+1)\times(n+1)$. The action of E on A involves either two rows or only one row. So we expand det(EA) across a row that is unchanged by the action of E, say, row i. Let A_{ij} (respectively, B_{ij}) be the matrix obtained by deleting row i and column j from A (respectively, B). Since these submatrices are only $n \times n$, the induction assumption implies that

$$detB_{ij} = \alpha \cdot detA_{ij}$$

where $\alpha = k, 1$, or -1, depending on the nature of E.

$$detEA = a_{i1}(-1)^{i+1}detB_{i1} + \cdots + a_{in}(-1)^{i+n}detB_{in}$$

$$= a_{i1}(-1)^{i+1}\alpha \cdot detA_{i1} + \cdots + a_{in}(-1)^{i+n}\alpha \cdot detA_{in}$$

$$= \alpha \cdot detA$$

6.3 Geometrical Interpretations of the Determinant

Fact 6.3.1
Determinant of an othogonal matrix is either 1 or -1.

Proof
We know that

$$A^T A = I_n$$

$$det(A^T A) = det(A^T)det(A) = det(A)^2 = 1$$

Fact 6.3.3
Consider a 2×2 matrix $A = [\vec{v_1}\vec{v_2}]$. Then, the area of the parallelogram defined by $\vec{v_1}$ and $\vec{v_2}$ is $|det(A)|$.

Proof

Consider the Gram-Schmidt process for two linearly independent vectors $\vec{v_1}$ and $\vec{v_2}$ in R^2. Let

$A = [\vec{v_1}\vec{v_2}]$
$B = [\vec{w_1}\vec{v_2}] \rightarrow det(B) = \frac{det(A)}{\|\vec{v_1}\|}$
$C = [\vec{w_1}\vec{w}] \rightarrow det(C) = det(B)$
$Q = [\vec{w_1}\vec{w_2}] \rightarrow det(Q) = \frac{det(C)}{\|\vec{w}\|}$

We conclude that

$$det(A) = \|\vec{v_1}\|\|\vec{v_2} - proj_{V_1}\vec{v_2}\|det(Q)$$

or

$$|det(A)| = \|\vec{v_1}\|\|\vec{v_2} - proj_{V_1}\vec{v_2}\|$$

If the direction of $\vec{v_2}$ is obtained by rotating $\vec{v_1}$ through a counterclockwise angle between 0 and π, then det(A)=$det[\ \vec{v_1}\ \ \vec{v_2}\ ]$ will be positive. If we rotate through a clockwise angle between 0 and $-\pi$, then det(A) will be negative.

Fact 6.3.5
Consider a 3×3 matrix $A = [\vec{v_1}\vec{v_2}\vec{v_3}]$. Then, the volume of the parallelepiped defined by $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ is $|det(A)|$.

Fact 6.3.4
If A is an $n \times n$ matrix with columns $\vec{v_1}, ..., \vec{v_n}$, then

$$|det(A)| = \|\vec{v_1}\|\|\vec{v_2} - proj_{V_1}\vec{v_2}\|...\|proj_{V_{n-1}}\vec{v_n}\|$$

Definition 6.3.6
Consider the vectors $\vec{v_1}, ..., \vec{v_k}$ in R^n. The k-volume of the k-parallelepiped defined by the vectors $\vec{v_1}, ..., \vec{v_k}$ is the set of all vectors in R^n of the form $c_1\vec{v_1} + c_2\vec{v_2} + ... + c_k\vec{v_k}$, where $0 \leq c_i \leq 1$. The k-volume $V(\vec{v_1}, ..., \vec{v_k})$ of this k-parallelepiped is defined recursively by $V(\vec{v_1}) = \|\vec{v_1}\|$ and

$$V(\vec{v_1}, ..., \vec{v_k}) = V(\vec{v_1}, ..., \vec{v}_{k-1})\|\vec{v_k} - proj_{V_{k-1}}\vec{v_k}\|$$

Fact 6.3.7
Consider the vectors $\vec{v_1}, ..., \vec{v_k}$ in R^n. Then the k-volume of the k-parallelepiped defined by the vectors $\vec{v_1}, ..., \vec{v_k}$ is

$$\sqrt{det(A^T A)}$$

where A is the $n \times k$ matrix with columns $\vec{v_1}, ..., \vec{v_k}$.

Proof
$A^T A = (QR)^T (QR) = R^T Q^T Q R = R^T R$,
because $Q^T Q = I_n$.

$$\begin{aligned} det(A^T A) &= det(R^T R) = det(R^T) det(R) \\ &= (r_{11} r_{22} ... r_{kk})^2 \\ &= (\|\vec{v_1}\| \|\vec{v_2} - proj_{V_1} \vec{v_2}\| ... \|\vec{v_k} - proj_{V_{k-1}} \vec{v_k}\|)^2 \\ &= (V(\vec{v_1}, ..., \vec{v_k}))^2 \end{aligned}$$

Fact 6.3.8 Expansion Factor
Consider a linear transformation $T(\vec{x}) = A\vec{x}$ from R^2 to R^2. Then, $|det(A)|$ is the expansion factor

$$\frac{area\ of\ T(\Omega)}{area\ of\ \Omega}$$

of T on parallelgrams Ω.
Likewise, for a linear transformation $T(\vec{x}) = A\vec{x}$ from R^n to R^n. Then, $|det(A)|$ is the expansion factor of T on n-parallelpipeds:

$$V(A\vec{v_1}, ..., A\vec{v_n}) = |det(A)|V(\vec{v_1}, ..., \vec{v_n}),$$

for all vectors $\vec{v_1}, ..., \vec{v_n}$ in R^n.

Using techniques of calculus, we can verify that $|det(A)|$ also gives the expansion factor of linear transformation T on any region Ω in the plane.

Fact 6.3.9 Cramer's Rule
Consider the linear system

$$A\vec{x} = \vec{b},$$

where A is an invertible $n \times n$ matrix. The components x_i of the solution vector $\vec{x}$ are

$$x_i = \frac{det(A_i(\vec{b}))}{det(A)},$$

where $A_i(\vec{b})$ is the matrix obtained by replacing the ith column of A by $\vec{b}$.

Proof
Write $A = [\ \vec{a_1}\ \ \vec{a_2}\ \ \ldots\ \ \vec{a_i}\ \ \ldots\ \ \vec{a_n}\ ]$. If $\vec{x}$ is the solution of the system $A\vec{x} = \vec{b}$, then

$$det(A_i(\vec{b})) = det[\ \vec{a_1}\ \ \vec{a_2}\ \ \ldots\ \ \vec{b}\ \ \ldots\ \ \vec{a_n}\ ]$$

$$= det[\ \vec{a_1}\ \ \vec{a_2}\ \ \ldots\ \ A\vec{x}\ \ \ldots\ \ \vec{a_n}\ ]$$

$$= det[\ \vec{a_1}\ \ \vec{a_2}\ \ \ldots\ \ (x_1\vec{a_1} + ... + x_i\vec{a_i} + ... + x_n\vec{a_n})\ \ \ldots\ \ \vec{a_n}\ ]$$

$$= det[\ \vec{a_1}\ \ \vec{a_2}\ \ \ldots\ \ x_i\vec{a_i}\ \ \ldots\ \ \vec{a_n}\ ]$$

$$= x_i det[\ \vec{a_1}\ \ \vec{a_2}\ \ \ldots\ \ \vec{a_i}\ \ \ldots\ \ \vec{a_n}\ ]$$

Note that we have used the linearity of the determinant in the ith column. Therefore,

$$x_i = \frac{det(A_i(\vec{b}))}{det(A)}.$$

Consider an invertible $n \times n$ matrix A and write

$$A^{-1} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1j} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2j} & \cdots & m_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nj} & \cdots & m_{nn} \end{bmatrix}$$

We know that $AA^{-1} = I_n$. Picking out the jth column of A^{-1}, we find that

$$A \begin{bmatrix} m_{1j} \\ m_{2j} \\ \vdots \\ m_{nj} \end{bmatrix} = \vec{e_j}$$

By Cramer's rule, $m_{ij} = det(A_i(e_j))/det(A)$, where the ith column of A is replaced by $\vec{e_j}$.

$$A_i(\vec{e_j}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & 0 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & 1 & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 & \cdots & a_{nn} \end{bmatrix}$$

Since $det(A_i(\vec{e_j}) = (-a)^{i+j} det(A_{ji})$, so that

$$m_{ij} = (-1)^{i+j} \frac{det(A_{ji})}{det(A)}.$$

Fact 6.3.10 Corallary to Cramer's rule
Consider an invertible $n \times n$ matrix A. The classical adjoint adj(A) is the $n \times n$ matrix whose ijth entry is $(-1)^{i+j} det(A_{ji})$. Then,

$$A^{-1} = \frac{1}{det(A)} adj(A) = \frac{1}{det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Example 5 Consider the linear system

$$ax + by = 1$$
$$cx + dy = 1$$

where $d > b > 0$ and $a > c > 0$. How does the solution x change as we change the parameters a and c? More precisely, find $\partial x/\partial a$ and $\partial x/\partial c$, and determine the signs of these quantities.

Solution

$$x = \frac{det \begin{bmatrix} 1 & b \\ 1 & d \end{bmatrix}}{det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \frac{d-b}{ad-bc}$$

$$\frac{\partial x}{\partial a} = \frac{-d(d-b)}{(ad-bc)^2} < 0$$

$$\frac{\partial x}{\partial c} = \frac{b(d-b)}{(ad-bc)^2} > 0$$

See Figure 9.

Example 6 For the vectors vectors $\vec{w_1}, \vec{w_2}$, and $\vec{b}$ shown in Figure 10, consider the linear system $A\vec{x} = \vec{b}$, where $A = [\ \vec{w_1} \ \ \vec{w_2}\ ]$. Cramer's rule tells us that

$$x_2 = \frac{det(A_2(\vec{b}))}{det(A)}$$

or

$$det(A_2(\vec{b})) = x_2 det(A)$$

Explain this geometrically, in terms of areas of parallelograms.