科目： **演 算 法** (Algorithms)

1. (25%) Given n objects with positive integer values $P_1, P_2, \ldots, P_n$ as profits, and positive integer values $W_1, W_2, \ldots, W_n$ as weights, along with a knapsack with positive integer value M as its capacity, the knapsack problem is to find a set of selected objects, each of which can be placed into the knapsack fractionally, such that the total profit of the selected objects is maximized while ensuring that the total weight of the selected objects does not exceed M. Write a greedy algorithm to solve the knapsack problem and analyze the time complexity of your algorithm.
2. (25%) The selection problem is the task of finding the k-th smallest element in an unsorted list of numbers. Given n numbers, the median-finding problem is a special case of the selection problem where $k=n/2$ (for even n) or $k=(n+1)/2$ (for odd n). Write a prune-and-search algorithm with $O(n)$ time complexity to solve the median-finding problem using the median-of-medians principle. You should analyze your algorithm to show its time complexity is indeed $O(n)$.
3. (25%) Show that the lower bound of the sorting problem that sorts n numbers is $\Omega(n \log n)$.
4. (25%) We have the following definitions and theorem related to NP-completeness.

   Definition 1. Let A_1 and A_2 be two problems. A_1 polynomially reduces to A_2 (written as $A_1 \propto A_2$) if and only if A_1 can be solved in polynomial time, by using a polynomial time algorithm which solves A_2.

   Definition 2. A problem is said to be a P (resp., NP) problem if it can be solved in polynomial time by a deterministic (resp., non-deterministic) algorithm.

   Definition 3. NP (resp., P) is the set of all NP (resp., P) problems.

   Definition 4. A problem A is NP-complete if $A \in$ NP and every NP problem polynomially reduces to A.

   Cook's Theorem. NP = P if and only if the satisfiability (SAT) problem is a P problem.

   (This implies that every NP problem polynomially reduces to SAT.)

   We assume that SAT $\propto$ B, B $\propto$ C, C $\propto$ D and D $\in$ NP. By the above assumptions, definitions, and theorem, show that D is NP-complete.