Optimization

Homework 1 Solutions

1. (a). We can rewrite $f$ as

$$f(x) = \frac{1}{2} x^T \begin{bmatrix} 2 & 6 \\ 6 & 14 \end{bmatrix} x + x^T \begin{bmatrix} 3 \\ 5 \end{bmatrix} + 6$$

The gradient and Hessian of $f$ are

$$\nabla f(x) = \begin{bmatrix} 2 & 6 \\ 6 & 14 \end{bmatrix} x + \begin{bmatrix} 3 \\ 5 \end{bmatrix},$$

$$F(x) = \begin{bmatrix} 2 & 6 \\ 6 & 14 \end{bmatrix}.$$  


(b). Suppose $d = [d_1, d_2]^T$ and $d_1^2 + d_2^2 = 1$

Then directional derivative $= \nabla f(x^T) d = [11,25] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 11 d_1 + 25 d_2$

Let $f' = 11 d_1 + 25 (1 - d_1^2)^{1/2}$

$$f'' = 11 + \frac{25}{2} (1 - d_1^2)^{-1/2} (-2 d_1) = 0$$

$$d_1 = \frac{11}{\sqrt{11^2 + 25^2}}, d_2 = \frac{25}{\sqrt{11^2 + 25^2}}.$$ Hence, $d = [\frac{11}{\sqrt{11^2 + 25^2}}, \frac{25}{\sqrt{11^2 + 25^2}}]^T$.

(c). The FONC in this case is $\nabla f(x) = 0$, The only point satisfying the FONC is $x^* = \frac{1}{2} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

The point above does not satisfy the SONC, because the Hessian is not positive definite (its determinant is negative). Therefore, $f$ does not have a minimizer.

2. (a). We can rewrite $f$ as

$$f(x) = \frac{1}{2} x^T \begin{bmatrix} 4 & 4 \\ 4 & 2 \end{bmatrix} x + x^T \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 7$$

The gradient and Hessian of $f$ are

$$\nabla f(x) = \begin{bmatrix} 4 & 4 \\ 4 & 2 \end{bmatrix} x + \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

$$F(x) = \begin{bmatrix} 4 & 4 \\ 4 & 2 \end{bmatrix}.$$
Hence \( \nabla f([0,1]^T) = [7,6]^T \).

directional derivative = \( \nabla f(x^T) \) d = 7

(b). The FONC in this case is \( \nabla f(x) = 0 \), The only point satisfying the FONC is 
\[
x^* = \frac{1}{4} \begin{bmatrix} -5 \\ 2 \end{bmatrix}
\]
\( f \) does not have a minimizer, because it does not satisfy SONC.

3. \( \nabla f(x) = [0,5]^T, \ d^T \nabla f(x) = 6 \ d_2, \) where \( d = [d_1, d_2]^T \)

(a). Because \( d_2 \) is allowed to be less than zero, it does not satisfy FONC.
(b). It satisfies SONC.
(c). \( x^* \) is not a local minimizer.

4. 
(a). \( x^* \) satisfies FONC.
(b). \( x^* \) does not satisfy SONC. ex: \( F(x) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \), if \( d_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), then \( d_1^T F d_1 > 0 \)
if \( d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), then \( d_2^T F d_2 < 0 \)
(c). \( x^* \) is not a local minimizer.

5. \( f'(x) = \nabla f(x) = 2x - 4 \sin x \).
\( f''(x) = F(x) = 2 - 4 \cos x \).

Using Newton’s method : 
\[
x^{(k+1)} = x^{(k)} - \frac{f'(x)}{f''(x)}
\]
\( x^{(1)} = -7.4727 ; x^{(2)} = 14.4785 ; x^{(3)} = 6.9351 ; x^{(4)} = 16.6354. \)

6. \( g'(x) = \nabla f(x) = 4(2x-1)+2^{14}(4-1024x)^3 \)

Using \( x^{(k)} = [g(x^{(k)}) x^{(k-1)} - g(x^{(k-1)}) x^{(k)}] / [g(x^{(k)}) - g(x^{(k-1)})] \) to calculate until \( |x^{(18)} - x^{(17)}| < x^{(17)} \times 10^{-5} \)
Hence, \( x^* \) is 0.0039671
\( g(x^*) = 0.9846. \)