Applied Linear Algebra

OTTO BRETSCHER

http://www.prenhall.com/bretscher

Chapter 8
Symmetric Matrices and Quadratic Forms

Chia-Hui Chang
Email: chia@csie.ncu.edu.tw
National Central University, Taiwan

8.1 SYMMETRIC MATRICES

In chapter 7, we are concerned with when is a given square matrix A diagonalizable? That is, when is there an `eigenbasis` for A?

In geometry, we prefer to work with `orthnomal bases`, which raises the question:

For which matrices is there an *orthonormal eigenbasis*?

Example 1 If A is orthogonally diagonalizable, what is the relationship between A^T and A?

Solution We have

$$S^{-1}AS = D$$

or

$$A = SDS^{-1} = SDS^T$$

for an orthogonal matrix S and a diagonal D. Then

$$A^T = (SDS^T)^T = SD^TS^T = SDS^T = A.$$

We find that A is symmetric.

Fact 8.1.1 Spectral theorem
A matrix A is orthogonally diagonalizable if and only if A is `symmetric` (i.e., $A^T = A$).

The set of eigenvalues of a matrix is called the spectrum of A, and the following description of the eigenvalues is called a spectral theorem.

THEOREM

The Spectral Theorem For A Symmetric Matrix

- A has n real eigenvalues, counting mutiplicities. (Fact 8.1.3)

- The dimension of the eigenspace for each eigenvalue λ equals the algebraic multiplicity of λ.

- The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal. (Fact 8.1.2)

- A is orthogonally diagonalizable. (Fact 8.1.1)

Example 2 For the symmetric matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$, find an orthogonal S such that $S^{-1}AS$ is diagonal.

Solution See Figure 1.

$$E_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, E_8 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

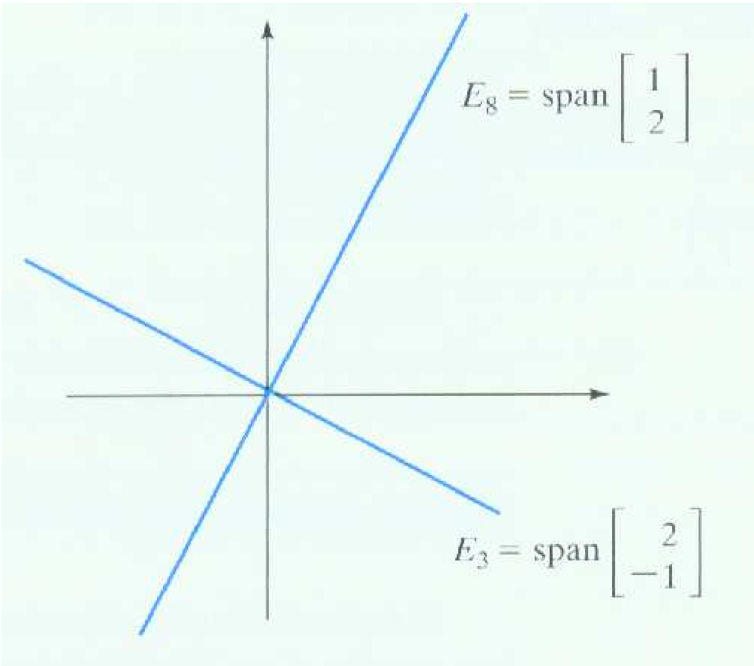

Figure 1

Note that the eigenspaces E_3 and E_8 are perpendicular. (This is no coincidence.) Therefore, we can find an orthonormal eigenbasis simply by dividing the given eigenvectors by their lengths:

$$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Define

$$S = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

then $S^{-1}AS = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$

Fact 8.1.2 Consider a symmetric matrix A. If $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A with distinct eigenvalues λ_1 and λ_2, then $\vec{v}_1 \cdot \vec{v}_2 = 0$; that is, $\vec{v}_2$ is orthogonal to $\vec{v}_1$.

Proof We compute the product $\vec{v}_1^T A \vec{v}_2$ in two ways:

- $\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T (\lambda_2 \vec{v}_2) = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2)$

- $\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T A^T \vec{v}_2 = (A\vec{v}_1)^T \vec{v}_2 = (\lambda_1 \vec{v}_1)^T \vec{v}_2 = \lambda_1 (\vec{v}_1 \cdot \vec{v}_2)$

Comparing the results, we find

$$\lambda_1 (\vec{v}_1 \cdot \vec{v}_2) = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2)$$

or

$$(\lambda_1 - \lambda_2)(\vec{v}_1 \cdot \vec{v}_2) = 0$$

Since $\lambda_1 \neq \lambda_2$, $\vec{v}_1 \cdot \vec{v}_2$ must be zero.

Fact 8.1.3 A symmetric $n \times n$ matrix A has n real eigenvalues if they are counted with their algebraic multiplicites.

Proof of 8.1.3 For those who have studied Section 7.5. Consider two complex conjugate eigenvalues $p \pm iq$ of A with corresponding eigenvectors $\vec{v} \pm i\vec{w}$. Compute the product

$$(\vec{v} + i\vec{w})^T A(\vec{v} - i\vec{w})$$

in two different ways:

$$(\vec{v} + i\vec{w})^T A(\vec{v} - i\vec{w}) = (\vec{v} + i\vec{w})^T (p - iq)(\vec{v} - i\vec{w})$$

$$= (p - iq)(\|\vec{v}\|^2 + \|\vec{w}\|^2)$$

$$(\vec{v} + i\vec{w})^T A(\vec{v} - i\vec{w}) = (A(\vec{v} + i\vec{w}))^T (\vec{v} - i\vec{w})$$

$$= (p+iq)(\vec{v}+i\vec{w})^T(\vec{v}-i\vec{w}) = (p+iq)(\|\vec{v}\|^2+\|\vec{w}\|^2)$$

Comparing the results, we find that $p + iq = p - iq$, so $q = 0$, as claimed.

Proof of 8.1.1 Even more technical.

Example 3 For the symmetric matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

find an orthogonal S such that $S^{-1}AS$ is diagonal.

Solution

The eigenvalues are 0 and 3, with

$E_0 = span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$ and $E_3 = span \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Note that the two eigenspaces are indeed perpendicular to one another (See Figure 2, 3).

We can construct an orthonormal eigenbasis for A by picking an orthonormal basis of each eigenspace.

Perform Gram-Schmidt process to the vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

we find

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

For E_3, we get

$$\vec{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, the orthogonal matrix

$$S = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

diagonalizes the matrix A:

$$S^{-1}AS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

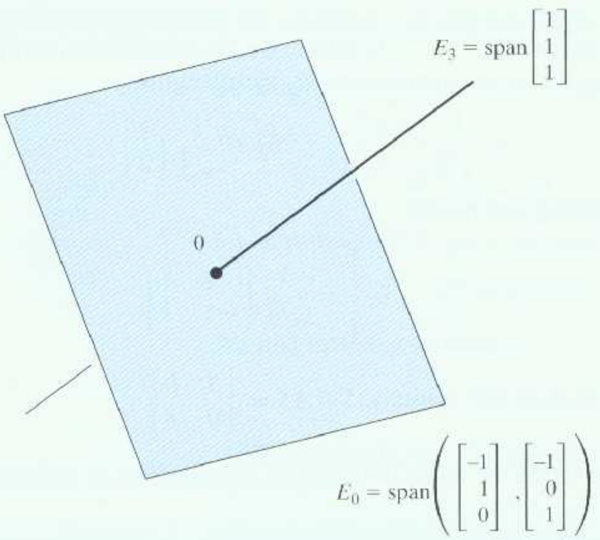

Figure 2 The eigenspaces E_0 and E_3 are orthogonal complements.

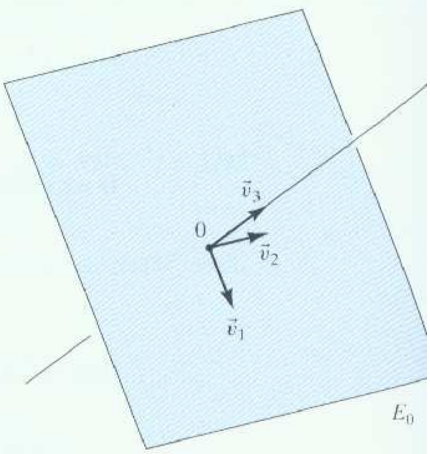

Figure 3

Algorithm 8.1.4 Orthogonal diagonalization of a symmetric matrix A

1. Find the eigenvalues of A, and find a basis of each eigenspace.

2. Using the Gram-Schmidt process, find an orthonormal basis of each eigenspace.

3. Form an orthonormal eigenbasis $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$ for A by combining the vectors you find in the last step, and let

$$P = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & ... & \vec{u}_n \\ | & | & & | \end{bmatrix}$$

P is orthogonal, and $P^{-1}AP$ will be diagonal.

Spectral Decomposition

Suppose that $A = PDP^{-1}$, where the columns of P are orthonormal eigenvectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n$ of A and the corresponding eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ are in the diagonal matrix D. Then, since $P^{-1} = P^T$,

$$A = PDP^T = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \vec{u}_1 & \cdots & \lambda_n \vec{u}_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix} = \lambda_1 \vec{u}_1 \vec{u}_1^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T$$

This representation of A is called a spectral decomposition of A because it breaks up A into pieces determined by the spectrum (eigenvalues) of A. Each term is an $n \times n$ matrix of rank 1. Furthermore, each matrix $\vec{u}_j \vec{u}_j^T$ is a projection matrix onto the subspace spanned by $\vec{u}_j$.

Example 4 Consider an invertible symmetric 2×2 matrix A. Show that the linear transformation $T(\vec{x} = A\vec{x}$ maps the unit circle into an ellipse, and find the lengths of the semimajor and the seminor axes of the ellipse in terms of the eigenvalues of A.

Solution

The spectral theorem tells us there is an orthonormal eigenbasis u_1, u_2 for T, with associated real eigenvalues λ_1, λ_2. Suppose that $|\lambda_1| > |\lambda_2|$. These eigenvalues will be nonzero, since A is invertible. The unit circle consists of all vectors of the form

$$\vec{v} = \cos(t)u_1 + \sin(t)u_2$$

. The image of the unit circle will be

$$T(\vec{v}) = \cos(t)T(u_1) + \sin(t)T(u_2)$$

$$= \cos(t)\lambda_1 u_1 + \sin(t)\lambda_2 u_2$$

an ellipse whose semimajor axis has the length $\|\lambda_1 u_1\| = |\lambda_1|$, while the length of the semiminor axis is $\|\lambda_2 u_2\| = |\lambda_2|$. (See Figure 4).