7.3 FINDING THE EIGENVECTORS OF A MATRIX

After we have found an eigenvalue λ of an $n \times n$ matrix A, we have to find the vectors $\vec{v}$ in R^n such that

$$A\vec{v} = \lambda\vec{v} \text{ or } (\lambda I_n - A)\vec{v} = \vec{0}$$

In other words, we have to find the *kernel* of the matrix $\lambda I_n - A$.

Definition 7.3.1 Eigenspace

Consider an eigenvalue λ of an $n \times n$ matrix A.Then the kernel of the matrix $\lambda I_n - A$ is called the $eigenspace$ associated with λ, denoted by E_λ:

$$E_\lambda = ker(\lambda I_n - A)$$

Note that E_λ consists of all solutions $\vec{v}$ of the linear system

$$A\vec{v} = \lambda\vec{v}$$

EXAMPLE 1 Let $T(\vec{x}) = A\vec{v}$ be the orthogonal projection onto a plane E in R^3. Describe the eigenspaces geometrically.

Solution See Figure 1.
The nonzero vectors $\vec{v}$ in E are eigenvectors with eigenvalue 1. Therefore, the eigenspace E_1 is just the plane E.

Likewise, E_0 is simply the kernel of A ($A\vec{v} = \vec{0}$); that is, the line $E^{\perp}$ perpendicular to E.

EXAMPLE 2 Find the eigenvectors of the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

Solution

See Section 7.2, Example 1, we saw the eigenvalues are 5 and -1. Then

$$E_5 = ker(5I_2 - A) = ker \begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix}$$
$$= ker \begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix} = span \begin{bmatrix} 2 \\ 4 \end{bmatrix} = span \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$E_{-1} = ker(-I_2 - A) = ker \begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix}$$
$$= span \begin{bmatrix} 2 \\ -2 \end{bmatrix} = span \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Both eigenspaces are lines, See Figure 2.

EXAMPLE 3 Find the eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution

Since

$$f_A(\lambda) = \lambda(\lambda - 1)^2$$

the eigenvalues are 1 and 0 with algebraic multiplicities 2 and 1.

$$E_1 = ker \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

To find this kernel, apply Gauss-Jordan Elimination:

$$\begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution of the system

$$\left| \begin{array}{lll} x_2 & & = 0 \\ & x_3 & = 0 \end{array} \right|$$

is

$$\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Therefore,

$$E_1 = span \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Likewise, compute the E_0:

$$E_0 = span \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Both eigenspaces are lines in the x_1-x_2-plane, as shown in Figure 3.

Compare with Example 1. There, too, we have two eigenvalues 1 and 0, but one of the eigenspace, E_1, is a plane.

Definition 7.3.2 Geometric multiplicity
Consider an eigenvalue λ if a matrix A. Then the dimension of eigenvalue $E_\lambda = ker(\lambda I_n - A)$ is called the *geometric multiplicity* of eigenvalue λ. Thus, the geometric multiplicity of λ is the *nullity* of matrix $\lambda I_n - A$.

Example 3 shows that the geometric multiplicity of an eigenvalue may be different from the algebraic multiplicity. We have

(algebraic multiplicity of eigenvalue 1)=2,

but

(geometric multiplicity of eigenvalue 1)=1.

Fact 7.3.3
Consider an eigenvalue λ of a matrix A. Then

(geometric multiplicity of λ)$\leq$
(algebraic multiplicity of λ).

EXAMPLE 4 Consider an upper triangular matrix of the form

$$A = \begin{bmatrix} 1 & \bullet & \bullet & \bullet & \bullet \\ 0 & 2 & \bullet & \bullet & \bullet \\ 0 & 0 & 4 & \bullet & \bullet \\ 0 & 0 & 0 & 4 & \bullet \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

What can you say about the geometric multiplicity of the eigenvalue 4?

Solution

$$E_4 = \begin{bmatrix} 3 & \bullet & \bullet & \bullet & \bullet \\ 0 & 2 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & \bullet & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \sharp & \bullet \\ 0 & 0 & 0 & 0 & \sharp \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The bullets on row 3 and 4 could be leading 1's. Therefore, the rank of this matrix will be between 2 and 4, and its nullity will be between 3 and 1. We can conclude that the geometric multiplicity of the eigenvalue 4 is less than the algebraic multiplicity.

Recall Fact 7.1.3, such a basis deserves a name.

Definition 7.3.4 Eigenbasis
Consider an $n\times$ n matrix A. A basis of R^n consisting of eigenvectors of A is called an $eigenbasis$ for A.

Example 1 Revisited: Projection on a plane E in R^3. Pick a basis $\vec{v}_1, \vec{v}_2$ of E and a nonzero $\vec{v}_3$ in $E^{\perp}$. The vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ form an eigenbasis. See Figure 4.

Example 2 Revisited: $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.
The vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ form an eigenbasis for A, see Figure 5.

Example 3 Revisited: $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.
There are not enough eigenvectors to form an eigenbasis. See Figure 6.

EXAMPLE 5 Consider a 3×3 matrix A with three eigenvalues, 1, 2, and 3. Let $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ be corresponding eigenvectors. Are vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ necessarily linearly independent?

Solution See Figure 7.
Consider the plane E spanned by $\vec{v}_1$, and $\vec{v}_2$. We have to examine $\vec{v}_3$ can not be contained in this plane.

Consider a vector $\vec{x} = c_1\vec{v_1} + c_2\vec{v_2}$ in E (with $c_1 \neq 0$ and $c_2 \neq 0$). Then $A\vec{x} = c_1A\vec{v}_1 + c_2A\vec{v}_2 = c_1\vec{v_1} + 2c_2\vec{v_2}$. This vector can not be a scalar multiple of $\vec{x}$; that is, E does not contain any eigenvectors besides the multiples of $\vec{v}_1$ and $\vec{v}_2$; in particular, $\vec{v}_3$ is not contained in E.

Fact 7.3.5 Considers the eigenvectors $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_m$ of an $n \times n$ matrix A, with distinct eigenvalues λ_1, λ_2, ..., λ_m. Then the $\vec{v}_i$ are linearly independent.

Proof

We argue by induction on m. Assume the claim holds for $m-1$. Consider a relation

$$c_1\vec{v}_1 + \cdots + c_{m-1}\vec{v}_{m-1} + c_m\vec{v}_m = \vec{0}$$

- apply the transformation A to both sides:

$$c_1\lambda_1\vec{v}_1 + \cdots + c_{m-1}\lambda_{m-1}\vec{v}_{m-1} + c_m\lambda_m\vec{v}_m = \vec{0}$$

- multiply both sides by λ_m:

$$c_1\lambda_m\vec{v}_1 + \cdots + c_{m-1}\lambda_m\vec{v}_{m-1} + c_m\lambda_m\vec{v}_m = \vec{0}$$

Subtract the above two equations:

$$c_1(\lambda_1-\lambda_m)\vec{v}_1 + \cdots + c_{m-1}(\lambda_{m-1}-\lambda_m)\vec{v}_{m-1} = \vec{0}$$

Since $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_{m-1}$ are linearly independent by induction, $c_i(\lambda_i - \lambda_m) = 0$, for $i = 1, ..., m-1$. The eigenvalues are assumed to be distinct; therefore $\lambda_i - \lambda_m \neq 0$, and $c_i = 0$. The first equation tells us that $c_m\vec{v}_m = \vec{0}$, so that $c_m = 0$ as well.

Fact 7.3.6 If an $n \times n$ matrix A has n distinct eigenvalues, then there is an eigenbasis for A. We can construct an eigenbasis by choosing an eigenvector for each eigenvalue.

EXAMPLE 6 Is there an eigenbasis for the following matrix?

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

Fact 7.3.7 Consider an $n \times n$ matrix A. If the geometric multiplicities of the eigenvalues of A add up to n, then there is an eigenbasis for A: We can construct an eigenbasis by choosing a basis of each eigenspace and combining these vectors.

Proof

Suppose the eigenvalues are λ_1, λ_2, ..., λ_m, with dim(E_{λ_i})=d_i. We first choose a basis $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_{d_1}$ of E_{λ_1}, and then a basis $\vec{v}_{d_1+1}$, ..., $\vec{v}_{d_1+d_2}$ of E_{λ_2}, and so on.

Consider a relation

$$\underbrace{c_1\vec{v}_1 + \cdots + c_{d_1}\vec{v}_{d_1}}_{\vec{w}_1 \text{ in } E_{\lambda_1}} + \underbrace{\cdots + c_{d_1+d_2}\vec{v}_{d_1+d_2}}_{\vec{w}_2 \text{ in } E_{\lambda_2}} + \cdots + \underbrace{\cdots + c_n\vec{v}_n}_{\vec{w}_m \text{ in } E_{\lambda_m}} = \vec{0}$$

Each under-braced sum $\vec{w}_i$ must be a zero vector since if they are nonzero eigenvectors, they must be linearly independent and the relation can not hold.

Because $\vec{w}_1 = 0$, it follows that $c_1 = c_2 = \cdots = c_{d_1} = 0$, since $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_{d_1}$ are linearly independent. Likewise, all the other c_j are zero.

EXAMPLE 7 Consider an Albanian mountain farmer who raises goats. This particular breed of goats has a life span of three years. At the end of each year t, the farmer conducts a census of his goats. He counts the number of young goats $j(t)$ (those born in the year t), the middle-aged ones $m(t)$ (born the year before), and the old ones $a(t)$ (born in the year $t-2$). The state of the herd can be represented by the vector

$$\vec{x}(t) = \begin{bmatrix} j(t) \\ m(t) \\ a(t) \end{bmatrix}$$

How do we expect the population to change from year to year? Suppose that for this breed and environment the evolution of the system can be modelled by

$$\vec{x}(t+1) = A\vec{x}(t)$$

where $A = \begin{bmatrix} 0 & 0.95 & 0.6 \\ 0.8 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$

We leave it as an exercise to interpret the entries of A in terms of reproduction rates and survival rates.

Suppose the initial populations are $j_0 = 750$ and $m_0 = a_0 = 200$.What will the populations be after t years, according to this model? What will happen in the long term?

Solution

Step 1: Find eigenvalues.

Step 2: Find eigenvectors.

Step 3: Express the initial vector $\vec{v}_0 = \begin{bmatrix} 750 \\ 200 \\ 200 \end{bmatrix}$ as a linear combination of eigenvectors.

Step 4: Write the closed formula for $\vec{v}(t)$.

Fact 7.3.8

The eigenvalues of similar matrices Suppose matrix A is similar to B. Then

1. Matrices A and B have the same characteristic polynomial; that is, $f_A(\lambda) = f_B(\lambda)$

2. rank(A) =rank(B) and nullity(A) =nullity(B)

3. Matrices A and B have the same eigenvalues, with the same algebraic and geometric multiplicities. (However,the eigenvectors need not be the same.)

4. det(A)=det(B) and tr(A)=tr(B)

Proof

a. If $B = S^{-1}AS$, then

$$f_B(\lambda) = det(\lambda I_n - B) = det(\lambda I_n - S^{-1}AS)$$

$$= det(S^{-1}(\lambda I_n - A)S) = det(S^{-1})det(\lambda I_n - A)det(S)$$

$$= det(\lambda I_n - A) = f_A(\lambda)$$

b. See Section 3.4, exercise 45 and 46.

c. If follows from part (a) that matrices A and B have the same eigenvalues, with the same algebraic multiplicities. As for for the geometric multiplicities, note that $\lambda I_n - A$ is similar to $\lambda I_n - B$ for all λ, so that nullity($\lambda I_n - A$)=nullity($\lambda I_n - B$) by part (b).

d. These equations follow from part (a) and Fact 7.2.5. Trance and determinant are coefficients of the characteristic polynomial.