Applied Linear Algebra

OTTO BRETSCHER

http://www.prenhall.com/bretscher

Chapter 7
Eigenvalues and Eigenvectors

Chia-Hui Chang
Email: chia@csie.ncu.edu.tw
National Central University, Taiwan

7.1 DYNAMICAL SYSTEMS AND EIGENVECTORS: AN INTRODUCTORY EXAMPLE

Consider a dynamical system:

$$x(t+1) = 0.86x(t) + 0.08y(t)$$

$$y(t+1) = -0.12x(t) + 1.14y(t)$$

Let

$$\vec{v}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

be the **state vector** of the system at time t.

We can write the matrix equation as

$$\vec{v}(t+1) = A\vec{v}(t)$$

where

$$A = \begin{bmatrix} 0.86 & 0.08 \\ -0.012 & 1.14 \end{bmatrix}$$

Suppose we know the initial state, we wish to find $\vec{v}(t)$, for any time t.

Case 1: Suppose $\vec{v}(0) = \begin{bmatrix} 100 \\ 300 \end{bmatrix}$

Case 2: Suppose $\vec{v}(0) = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$

Case 3: Suppose $\vec{v}(0) = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$

Case 1:

$$\vec{v}(1) = A\vec{v}(0) = \begin{bmatrix} 0.86 & 0.08 \\ -0.012 & 1.14 \end{bmatrix} \begin{bmatrix} 100 \\ 300 \end{bmatrix} = \begin{bmatrix} 110 \\ 330 \end{bmatrix}$$

$\vec{v}(1) = A\vec{v}(0) = 1.1\vec{v}(0)$
$\vec{v}(2) = A\vec{v}(1) = A(1.1\vec{v}(0)) = 1.1^2\vec{v}(0)$
$\vec{v}(3) = A\vec{v}(2) = A(1.1^2\vec{v}(0)) = 1.1^3\vec{v}(0)$
$\vdots$
$\vec{v}(t) = 1.1^t\vec{v}(0)$

Case 2:

$$\vec{v}(1) = A\vec{v}(0) = \begin{bmatrix} 0.86 & 0.08 \\ -0.012 & 1.14 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 180 \\ 90 \end{bmatrix}$$

$\vec{v}(1) = A\vec{v}(0) = 0.9\vec{v}(0)$
$\vec{v}(t) = 0.9^t\vec{v}(0)$

Case 3:

$$\vec{v}(1) = A\vec{v}(0) = \begin{bmatrix} 0.86 & 0.08 \\ -0.012 & 1.14 \end{bmatrix} \begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 940 \\ 1020 \end{bmatrix}$$

The state vector $\vec{v}(1)$ is not a scalar multiple of the initial state $\vec{v}(0)$. We have to look for another approach.

Consider the two vectors

$$\vec{v}_1 = \begin{bmatrix} 100 \\ 300 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

Since the system is linear and

$$\vec{v}(0) = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = 2\vec{v}_1 + 4\vec{v}_2$$

Therefore,

$$\vec{v}(t) = A^t\vec{v}(0) = A^t(2\vec{v}_1 + 4\vec{v}_2) = 2A^t\vec{v}_1 + 4A^t\vec{v}_2$$
$$= 2(1.1)^t\vec{v}_1 + 4(0.9)^t\vec{v}_2$$
$$= 2(1.1)^t \begin{bmatrix} 100 \\ 300 \end{bmatrix} + 4(0.9)^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

$$x(t) = 200(1.1)^t + 800(0.9)^t$$

$$y(t) = 600(1.1)^t + 400(0.9)^t$$

Since the terms involving 0.9^t approach zero as t increases, $x(t)$ and $y(t)$ eventually grow by about 10% each time, and their ratio $y(t)/x(t)$ approaches 600/200=3.

See Figure 3, The state vector $\vec{x}(t)$ approaches the line L_1, with the slope 3.

Connect the tips of the state vector $\vec{v}(i), i = 1, 2, ..., t$, the trajectory is shown in Figure 4.

Sometimes, we are interested in the state of the system in the past at times -1, -2, ...

For different $\vec{v}(0)$, the trajectory is different. Figure 5 shows the trajectory that starts above L_1 and one that starts below L_2.

From a mathematical point of view, it is informative to sketch a phase portrait of this system in the whole $c-r$-plane (see Figure 6), even though the trajectories outside the first quadrant are meaningless in terms of population study.

Eigenvectors and Eigenvalues
How do we find the initial state vector $\vec{v}$ such that $A\vec{v}$ is a scalar multiple of $\vec{v}$, or

$$A\vec{v} = \lambda\vec{v},$$

for some scalar λ?

Definition 7.1.1
Eigenvectors and eigenvalues Consider an $n \times n$ matrix A. A **nonzero** vector $\vec{v}$ in R^n is called an $eigenvector$ of A if $A\vec{v}$ is a scalar multiple of $\vec{v}$, that is, if

$$A\vec{v} = \lambda\vec{v}$$

for some scalar λ. Note that this scalar λ may be zero. The scalar λ is called the $eigenvalue$ associated with the eigenvector $\vec{v}$.

EXAMPLE 1
Find all eigenvectors and eigenvalues of the identity matrix I_n.

Solution All nonzero vectors in R^n are eigenvectors, with eigenvalue 1.

EXAMPLE 2
Let T be the orthogonal projection onto a line L in R^2. Describe the eigenvectors of T geometrically and find all eigenvalues of T.

Solution (See Figure 8.) (a). Any vector $\vec{v}$ on L is a eigenvector with eigenvalue 1. (b). Any vector $\vec{w}$ perpendicular to L is a eigenvector with eigenvalue 0.

EXAMPLE 3
Let T from R^2 to R^2 be the rotation in the plane through an angle of 90° in the counter-clockwise direction. Find all eigenvalues and eigenvectors of T. (See Figure 9)

Solution There are no eigenvectors and eigenvalues here.

EXAMPLE 4
What are the possible real eigenvalues of an orthogonal matrix A?

Solution The possible real eigenvalue is 1 or -1 since orthogonal transformation preserves length.

Dynamical Systems and Eigenvectors

Fact 7.1.3 Discrete dynamical systems
Consider the dynamical system

$$\vec{x}(t+1) = A\vec{x}(t) \text{ with } \vec{x}(0) = \vec{x}_0$$

Then

$$\vec{x}(t) = A^t\vec{x}_0$$

Suppose we can find a basis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ of R^n consisting of eigenvectors of A with

$$A\vec{v}_1 = \lambda_1\vec{v}_1, A\vec{v}_2 = \lambda_2\vec{v}_2, \ldots, A\vec{v}_n = \lambda_n\vec{v}_n.$$

Find the coordinates $c_1, c_2, \ldots, c_n$ of vector $\vec{x}_0$ with respect to $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ of R^n:

$$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ . \\ c_n \end{bmatrix}$$

Let $S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}$.

Then $\vec{x}_0 = S \begin{bmatrix} c_1 \\ c_2 \\ . \\ c_n \end{bmatrix}$ so that $\begin{bmatrix} c_1 \\ c_2 \\ . \\ c_n \end{bmatrix} = S^{-1}\vec{x}_0$.

Consider

$$\vec{x}(t) = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + \cdots + c_n\lambda_n^t\vec{v}_n.$$

We can write this equation in matrix form as

$$\vec{x}(t) = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1^t & 0 & . & 0 \\ 0 & \lambda_2^t & 0 & 0 \\ . & . & . & . \\ 0 & 0 & 0 & \lambda_n^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ . \\ c_n \end{bmatrix}$$

$$= S \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & 0 & 0 \\ . & . & . & . \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}^t S^{-1}\vec{x}_0$$

Definition 7.1.4

Discrete trajectories and phase portraits

Consider a discrete dynamical system

$$\vec{x}(t+1) = A\vec{x}(t)$$

with initial value $\vec{x}(0) = \vec{x}_0$ where A is a 2×2 matrix. In this case, the state vector $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ can be represented geometrically in the $x_1 - x_2$-plane.

The endpoints of state vectors $\vec{x}(0) = \vec{x}_0$, $\vec{x}(1) = A\vec{x}_0$, $\vec{x}(2) = A^2\vec{x}_0$, . . . form the (discrete) *trajectory* of this system, representing its evolution in the future. Sometimes we are interested in the past states $\vec{x}(-1) = A^{-1}\vec{x}_0$, $\vec{x}(-2) = (A^2)^{-1}\vec{x}_0$, . . . as well. It is suggestive to "connect the dots" to create the illusion of a continuous trajectories. Take another look at Figure 4.

A (discrete) *phase portrait* of the system $\vec{x}(t+1) = A\vec{x}(t)$ shows discrete trajectories for various initial states, capturing all the qualitatively different scenarios (as in Figure 6).

See Figure 11, we sketch phase portraits for the case when A has two eigenvalues $\lambda_1 > \lambda_2 > 0$. (Leave out the special case when one of the eigenvalues is 1.) Let $L_1 = span(\vec{v}_1)$ and $L_2 = span(\vec{v}_2)$. Since

$$\vec{x}(t) = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2$$

we can sketching the trajectories for the following cases:

(a) $\lambda_1 > \lambda_2 > 1$

(b) $\lambda_1 > 1 > \lambda_2$

(c) $1 > \lambda_1 > \lambda_2$

Summary 7.1.4

Consider an $n \times n$ matrix

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

Then the following statements are equivalent:

i. A is invertible.

ii. The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, for all $\vec{b}$ for all $\vec{b}$ in R^n.

iii. $rref(A) = I_n$.

iv. $rank(A) = n$.

v. $im(A) = R^n$.

vi. $ker(A) = \{\vec{0}\}$.

vii. The $\vec{v}_i$ are a basis of R^n.

viii. The $\vec{v}_i$ span R^n.

ix. The $\vec{v}_i$ are linearly independent.

x. $det(A) \neq 0$.

xi. 0 fails to be an eigenvalue of A.