5.2 GRAM-SCHMIDT PROCESS AND QR FACTORIZATION

How can we construct an orthonormal basis? Say, from any basis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ of a subspace V?

If V is a line with basis $\vec{v}_1$:

$$\vec{w}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1$$

When V is a plane with basis $\vec{v}_1, \vec{v}_2$, we first get $\vec{w}_1$ as above.

Next find a vector in V orthogonal to $\vec{w}_1$.

$$\vec{v}_2 - proj_L\vec{v}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1$$

Then Divide the vector by its length to get the second vector $\vec{w}_2$.

$$\vec{w}_2 = \frac{1}{\|\vec{v}_2 - proj_L\vec{v}_2\|}(\vec{v}_2 - proj_L\vec{v}_2)$$

See Figure 1, 2, 3.

EXAMPLE 1 Find an orthonormal basis of the subspace

$$V = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix}\right)$$

of R^4, with basis

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix}.$$

Solution

Using the terminology just introduced, we find the following results:

$$\| \vec{v_1} \| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2,$$

$$\vec{w_1} = \frac{1}{\|\vec{v_1}\|}\vec{v_1} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

$$\vec{w_1} \cdot \vec{v_2} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix} = 10,$$

$$proj_L \vec{v_2} = (\vec{w_1} \cdot \vec{v_2})\vec{w_1} = 10 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$$

$$\vec{v_2} - proj_L \vec{v_2} = \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix}.$$

$$\| \vec{v_2} - proj_L \vec{v_2} \| = \sqrt{4 \cdot 16} = 8,$$

$$\vec{w_2} = \frac{1}{\|\vec{v_2} - proj_L\vec{v_2}\|}(\vec{v_2} - proj_L\vec{v_2})$$

$$= \frac{1}{8}\begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

We have found an orthonormal basis of V:

$$\vec{w_1} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \vec{w_2} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

We can represent the preceding computations more succinctly in matrix form. Let's solve the equations defining $\vec{w_1}$ and $\vec{w_2}$.

$\vec{w_1} = \frac{1}{\|\vec{v_1}\|}\vec{v_1}$ and $\vec{w_2} = \frac{1}{\|\vec{v_2} - proj_L\vec{v_2}\|}(\vec{v_2} - proj_L\vec{v_2})$, for vectors $\vec{v_1}$ and $\vec{v_2}$:

$$\vec{v_1} = \| \vec{v_1} \| \vec{w_1},$$

and

$$\vec{v_2} = proj_L \vec{v_2} + \| \vec{v_2} - proj_L \vec{v_2} \| \vec{w_2}$$

$$= (\vec{w_1} \cdot \vec{v_2}) \vec{w_1} + \| \vec{v_2} - proj_L \vec{v_2} \| \vec{w_2}.$$

We can write the last two equations in matrix form:

$$\begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{w_1} & \vec{w_2} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \| \vec{v_1} \| & \vec{w_1} \cdot \vec{v_2} \\ 0 & \| \vec{v_2} - proj_L \vec{v_2} \| \end{bmatrix}}_{R}$$

Note that we have written 4×2 matrix Q with orthonormal columns and the upper triangular 2×2 matrix R with positive entries on the diagonal.

Matrix Q stores the orthonormal basis $\vec{w_1}$, $\vec{w_2}$ we constructed, and matrix R gives the relationship between the "old" basis $\vec{v_1}$, $\vec{v_2}$, and the "new" basis $\vec{w_1}$, $\vec{w_2}$ of V.

Let's plug in numbers (note that we computed all the entries of matrix of matrix R in the process of finding $\vec{w_1}$ and $\vec{w_2}$):

$$\begin{bmatrix} 1 & 1 \\ 1 & 9 \\ 1 & 9 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 10 \\ 0 & 8 \end{bmatrix}$$

Algorithm 5.2.1
The Gram-Schmidt process

Consider a subspace V of R^n with basis $\vec{v_1}$,$\vec{v_2}$,...,$\vec{v_m}$. We wish to construct an orthonormal basis $\vec{w_1}$,$\vec{w_2}$,...,$\vec{w_m}$ of V.

Let $\vec{w_1} = (\frac{1}{\|\vec{v_1}\|})\vec{v_1}$. As we define $\vec{w_j}$ for $j = 2, 3, ..., m$, we may assume that an orthonormal basis $\vec{w_1}$,$\vec{w_2}$,...,$\vec{w_{j-1}}$ of $V_{j-1} = span(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{j-1}})$ has already been constructed. Let

$$\vec{w_j} = \frac{1}{\|\vec{v_j} - proj_{V_{j-1}}\vec{v_j}\|}(\vec{v_j} - proj_{V_{j-1}}\vec{v_j}).$$

Note that

$$proj_{V_{j-1}}\vec{v_j}$$

$$= (\vec{w_1} \cdot \vec{v_j})\vec{w_1} + (\vec{w_2} \cdot \vec{v_j})\vec{w_2} + \ldots + (\vec{w_{j-1}} \cdot \vec{v_j})\vec{w_{j-1}},$$

by Fact 5.1.6.

THE QR Factorization

The Gram-Schmidt process can be presented succinctly in matrix form, as illustrated in Example 1. Using the terminology introduced in Algorithm 5.2.1, we can write

$$\vec{v_1} = \|\vec{v_1}\|\vec{w_1}$$

and

$$\vec{v_j} = proj_{V_{j-1}}\vec{v_j} + \|\vec{v_j} - proj_{V_{j-1}}\vec{v_j}\|\vec{w_j}$$

$$= (\vec{w_1}\vec{v_j})\vec{w_1} + \cdots + (\vec{w}_{j-1}\vec{v_j})\vec{w}_{j-1} + \|\vec{v_j} - proj_{V_{j-1}}\vec{v_j}\|\vec{w_j}$$

(for j=2,3,...,m).

Let

$$\begin{aligned} r_{11} = &\ \|\vec{v_1}\| & \\ r_{jj} = &\ \|\vec{v_j} - proj_{V_{j-1}}\vec{v_j}\| & (j = 2, 3, ..., m), \\ r_{ij} = &\ \vec{w_i} \cdot \vec{v_j} & (i < j). \end{aligned}$$

Then,

$$\begin{aligned}
&\vec{v_1} = r_{11}\vec{w_1} \\
&\vec{v_2} = r_{12}\vec{w_1} + r_{22}\vec{w_2} \\
&\vdots \\
&\vec{v_m} = r_{1m}\vec{w_1} + r_{2m}\vec{w_2} + \cdots + r_{mm}\vec{w_m}.
\end{aligned}$$

We can write these equations in matrix form:

$$\begin{bmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_m} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{w_1} & \vec{w_2} & \cdots & \vec{w_m} \\ | & | & & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ 0 & r_{22} & \cdots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{mm} \end{bmatrix}$$

$$M = QR$$

Note that M is an $n \times m$ matrix with linearly independent columns, Q is an $n \times m$ matrix with orthonormal columns, and R is an upper triangular $m \times m$ matrix with positive entires on the diagonal.

Fact 5.2.2 QR factorization

Consider an $n \times m$ matrix M with linearly independent columns $\vec{v_1}, ..., \vec{v_m}$. Then there is an $n \times m$ matrix Q whose columns $\vec{w_1}, ..., \vec{w_m}$ are orthonormal and an upper triangular $m \times m$ matrix R with positive diagonal entries such that

$$M = QR.$$

This representation is unique. Furthermore, $r_{11} = \|\vec{v_1}\|$, $r_{ij} = \|\vec{v_j} - proj_{V_{j-1}}\vec{v_j}\|$(for $j > 1$),

and $r_{ij} = \vec{w_i} \cdot \vec{v_j}$ (for $i < j$),

where $V_{j-1} = span(\vec{v_1}, \vec{v_2}, ..., \vec{v}_{j-1})$.

EXAMPLE 2 Find the QR factorization of the shear matrix M= $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Solution

Here

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

As in Example 1, the QR factorization of M will have the form

$$M = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} = \begin{bmatrix} \vec{w_1} & \vec{w_2} \end{bmatrix} \begin{bmatrix} \| \vec{v_1} \| & \vec{w_1} \cdot \vec{v_2} \\ 0 & \| \vec{v_2} - proj_{V_1}\vec{v_2} \| \end{bmatrix}$$

We will compute the columns of W and the entries of R step by step:

$$r_{11} = \|\vec{v_1}\| = \sqrt{2}$$

$$\vec{w_1} = \frac{1}{\|\vec{v_1}\|}\vec{v_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$r_{12} = \vec{w_1} \cdot \vec{v_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$\vec{v_2} - proj_{v_1}\vec{v_2} = \vec{v_2} - (\vec{w_1} \cdot \vec{v_2})\vec{w_1}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

$$r_{22} = \|\vec{v_2} - proj_{v_1}\vec{v_2}\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$$

$$\vec{w_2} = \frac{1}{\|\vec{v_2} - proj_{v_1}\vec{v_2}\|}(\vec{v_2} - proj_{v_1}\vec{v_2})$$

$$= \sqrt{2} \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = M = QR = \begin{bmatrix} \vec{w_1} & \vec{w_2} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$$

$$= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \right).$$

Draw pictures analogous to Figures 1 through 3 to illustrate these computations!

Exercise 5.2 5, 11, 13, 19, 27, 31, 33, 37