4.3 COORDINATES IN A LINEAR SPACE

By introducing coordinates, we can transform any n-dimensional linear space into R^n

4.3.1 Coordinates in a linear space

Consider a linear space V with a basis B consisting of $f_1, f_2, ...f_n$. Then any element f of V can be written uniquely as

$$\text{f} = c_1f_1 + c_2f_2 + \cdots + c_nf_n,$$

for some scalars $c_1, c_2, ..., c_n$. There scalars are called the B coordinates of f, and the vector

$$\begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{bmatrix}$$

is called the B-coordinate vector of f, denoted by $[f]_B$.

The B coordinate transformation $T(f) = [f]_B$ from V to R^n is an isomorphism (i.e., an invertible linear transformation). Thus, V is isomorphic to R^n; the linear spaces V and R^n have the same structure.

Example. *Choose a basis of P_2 and thus transform P_2 into R^n, for an appropriate n.*

Example. *Let V be the linear space of upper-triangular 2×2 matrices (that is, matrices of the form*

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}.$$

Choose a basis of V and thus transform V into R^n, for an appropriate n.

Example. *Do the polynomials,* $f_1(x) = 1 + 2x + 3x^2$*,* $f_2(x) = 4 + 5x + 6x^2$*,* $f_3(x) = 7 + 8x + 10x^2$ *from a basis of* P_2*?*

Solution

Since P_2 is isomorphic to R^3, we can use a coordinate transformation to make this into a problem concerning R^3. The three given polynomials form a basis of P_2 if the coordinate vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

form a basis of R^3.

Fact Two bases of a linear space consists of the same number of elements.

Proof Suppose two bases of a linear space V are given: basis $\coprod$, consisting of $f_1, f_2, \ldots, f_n$ and basis $\Im$ with m elements. We need to show that $m = n$.

Consider the vectors $[f_1]_\Im, [f_2]_\Im, \ldots, [f_n]_\Im$, these n vectors form a basis of R^m, since the $\Im$-coordinate transformation is an isomorphism from V to R^m.

Since all bases of R^m consist of m elements, we have $m = n$, as claimed.

Example. *Consider the linear transformation*

$$T\ (f) = f' + f'' \text{ form } P_2 \text{ to } P_2.$$

Since P_2 is isomorphic to R^3, this is essentially a linear transformation from R^3 to R^3, given by a 3×3 matrix B. Let's see how we can find this matrix.

Solution

We can write transformation T more explicitly as

$$\begin{aligned} T\ (a + bx + cx^2) &= (b + 2cx) + 2c \\ &= (b + 2c) + 2cx. \end{aligned}$$

Next let's write the input and the output of T in coordinates with respect to the standard basis B of P_2 consisting of $1, x, x^2$:

$$a + bx + cx^2 \longrightarrow (b + 2c) + 2cx$$

See Figure 1

Written in B coordinates, transformation T takes $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ into $\begin{bmatrix} b+2c \\ 2c \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

The matrix $B = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ is called the matrix of T. It describes the transformation T if input and output are written in B coordinates.
Let us summarize our work in a diagram:

See Figure 2

Definition 4.3.2 B-Matrix of a linear transformation
Consider a linear transformation T from V to V, where V is an n-dimensional linear space. Let B be a basis of V. Then, there is an $n \times n$ matrix B that transform [f]$_B$ into $[T(f)]_B$, called the B-matrix of T.

$$[T(f)]_B = B[f]_B$$

Fact 4.3.3 The columns of the B-matrix of a linear transformation

Consider a linear transformation T from V to V, and let B be the matrix of T with respect to a basis B of V consisting of $f_1, \ldots, f_n$.
Then

$$B = [[T(f_1)] \cdots [T(f_n)]] \, .$$

That is, the columns of B are the B-coordinate vectors of the transformation of the basis elements.

Proof

If

$$f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n,$$

then

$$T(f) = c_1 T(f_1) + c_2 T(f_2) + \cdots + c_n T(f_n),$$

and

$$[T(f)]_B = c_1[T(f_1)]_B + c_2[T(f_2)]_B + \cdots + c_n[T(f_n)]_B$$

$$= \left[\ [T(f_1)]_B \ \ \cdots \ \ [T(f_n)]_B \ \right] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$= \left[\ [T(f_1)]_B \ \ \cdots \ \ [T(f_n)]_B \ \right] [f]_B$$

Example. *Use Fact 4.3.3 to find the matrix B of the linear transformation*

$$T(f) = f' + f'' \text{ from } P_2 \text{ to } P_2$$

with respect to the standard basis B *(See Example 4.)*

Solution

$$B = \left[\ [T(1)]_B \ \ [T(x)]_B \ \ [T(x^2)]_B\ \right]$$

$$B = \left[\ [0]_B \ \ [1]_B \ \ [2+2x]_B\ \right]$$

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Example. *Consider the function*

$$T(M) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M - M \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

from $R^{2\times 2}$ *to* $R^{2\times 2}$*. We are told that* T *is a linear transformation.*

1. *Find the matrix* B *of* T *with respect to the standard basis* B *of* $R^{2\times 2}$
   (Hint: use column by column or definition)

2. *Find image and kernel of B.*

3. *Find image and kernel of T.*

4. *Find rank and nullity of transformation T.*

Solution

a. Use definition

$$T(M) = T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ a & c \end{bmatrix} = \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix}$$

Now we write input and output in B-coordinate:

See Figure 3

We can see that

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

b. To find the kernel and image of matrix B, we compute rref(B) first:

$$rref(B) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ is a basis of im(B)

and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis of ker(B).

c. To find image of kernel of T, we need to transform the vectors back into $R^{2\times 2}$:

$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a basis of im(B)

and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a basis of ker(B).

d.

$$rank(T) = dim(imT) = 2$$

and

$$nullity(T) = dim(kerT) = 2.$$

Fact 4.3.4 The matrices of T with respect to different bases

Suppose that A and B are two bases of a linear space V and that T a linear transformation from V to V.

1. There is an invertible matrix S such that $[f]_A = S[f]_B$ for all f in V.

2. Let A and B be the B-matrix of T for these two bases, respectively. Then matrix A is $similar$ to B. In fact, B = $S^{-1}AS$ for the matrix S from part(a).

Proof

a. Suppose basis B consists of $f_1, f_2, \ldots, f_n$. If

$$f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n,$$

then

$$[f]_A = [c_1 f_1 + c_2 f_2 + \cdots + c_n f_n]_A$$

$$= c_1[f_1]_A + c_2[f_2]_A + \cdots + c_n[f_n]_A$$

$$= \begin{bmatrix} [f_1]_A & [f_2]_A & \cdots & [f_n]_A \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} [f_1]_A & [f_2]_A & \cdots & [f_n]_A \end{bmatrix}}_{S} [f]_B$$

b. Consider the following diagram:

See Figure 4.

Performing a "diagram chase," we see that

$$AS = SB, \text{ or } B = S^{-1}AS.$$

See Figure 5.

Example. *Let V be the linear space spanned by functions e^x and e^{-x}. Consider the linear transformation D(f) = f' from V to V:*

1. *Find the matrix A of D with respect to basis B consisting of e^x and e^{-x}.*

2. *Find the matrix B of D with respect to basis B consisting of $(\frac{1}{2}(e^x+e^{-x}))$ and $(\frac{1}{2}(e^x-e^{-x}))$. (These two functions are called the $hypeerboliccosine$, cosh(x), and the $hypeerbolicsine$, sinh(x), respectively.)*

3. *Using the proof of Fact 4.3.4 as a guide, construct a matrix S such that B $= S^{-1}AS$, showing that matrix A is similar to B.*

Exercise 4.3: 3, 7, 9, 13, 21, 34, 35, 37

Example Let V be the linear space of all functions of the form $f(x) = a\cos(x) + b\sin(x)$, a subspace of C^∞. Consider the transformation

$$T(f) = f'' - 2f' - 3f$$

from V to V.

1. Find the matrix B of T with respect to the basis B consisting of functions cos(x) and sin(x).

2. Is T an isomorphism?

3. How many solutions f in V does the differential equation

   $$f''(x) - 2f'(x) - 3f(x) = cos(x)$$

   have?