4.2 LINEAR TRANSFORMATIONS AND ISOMORPHISMS

Definition 4.2.1

Linear transformation Consider two linear spaces V and W. A function T from V to W is called a linear transformation if:

$$T(f+g) = T(f) + T(g) \text{ and } T(kf) = kT(f)$$

for all elements f and g of V and for all scalar k.

Image, Kernel For a linear transformation T from V to W, we let

$$im(T) = \{T(f) : f \in V\}$$

and

$$ker(T) = \{f \in V : T(f) = 0\}$$

Note that $im(T)$ is a subspace of co-domain W and $ker(T)$ is a subspace of domain V.

Rank, Nullity

If the image of T is finite-dimensional, then $dim(imT)$ is called the rank of T, and if the kernel of T is finite-dimensional, then $dim(kerT)$ is the nullity of T.

If V is finite-dimensional, then the rank-nullity theorem holds (see fact 3.3.9):

$$\begin{aligned} \text{dim(V)} &= \text{rank(T)+nullity(T)} \\ &= \text{dim(imT)+dim(kerT)} \end{aligned}$$

Definition 4.2.2 Isomorphisms and isomorphic spaces

An invertible linear transformation is called an $isomorphism$. We say the linear space V and W are isomorphic if there is an isomorphism from V to W.

EXAMPLE 4 Consider the transformation

$$T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

from R^4 to $R^{2\times 2}$.
We are told that T is a linear transformation.
Show that transformation T is invertible.

Solution

The most direct way to show that a function is invertible is to find its inverse. We can see that

$$T^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

The linear spaces R^4 and $R^{2\times 2}$ have essentially the same structure. We say that the linear spaces R^4 and $R^{2\times 2}$ are *isomorphic*.

EXAMPLE 5 Show that the transformation

$$T(A) = S^{-1}AS \text{ from } R^{2\times 2} \text{ to } R^{2\times 2}$$

is an isomorphism, where $S = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution

We need to show that T is a linear transformation, and that T is invertible.

Let's think about the linearity of T first:

$$T(M+N) = S^{-1}(M+N)S = S^{-1}(MS+NS)$$

$$= S^{-1}MS + S^{-1}NS$$

equals $T(M) + T(N) = S^{-1}MS + S^{-1}NS$ and

$$T(kA) = S^{-1}(kA)S = k(S^{-1}AS)$$

equals $kT(A) = k(S^{-1}AS)$.

The inverse transformation is

$$T^{-1}(B) = SBS^{-1}$$

Fact 4.2.3 Properties of isomorphisms

1. If T is an isomorphism, then so is T^{-1}

2. A linear transformation T from V to W is an isomorphism if (and only if)

$$ker(T) = \{0\}, im(T) = W$$

3. Consider an isomorphism T from V to W.If
$f_1, f_2, ...f_n$
is a basis of V, then $T(f_1), T(f_2), ...T(f_n)$ is a basis of W.

4. If V and W are isomorphic and dim(V)=n, then dim(W)=n.

Proof

1. We must show that T^{-1} is linear. Consider two elements f and g of the codomain of T:

$$T^{-1}(f+g) = T^{-1}(TT^{-1}(f) + TT^{-1}(g))$$

$$= T^{-1}(T(T^{-1}(f) + T^{-1}(g)))$$

$$= T^{-1}(f) + T^{-1}(g)$$

   In a similar way, you can show that $T^{-1}(kf) = kT^{-1}(f)$, for all f in the codomain of T and all scalars k.

2. $\Rightarrow$ To find the kernel of T, we have to solve the equation
   $T(f) = 0$, Apply T^{-1} on both sides
   $T^{-1}T(f) = T^{-1}(0), \rightarrow f = T^{-1}(0) = 0$
   so that $ker(T) = 0$, as claimed.

Any g in W can be written as $g = T(T^{-1}(g))$, so that $im(T) = W$.

$\Leftarrow$ Suppose $ker(T) = \{0\}$ and $im(T) = W$. We have to show that T is invertible, i.e. the equation $T(f) = g$ has a unique solution f for any g in W.
There is at last one such solution, since $im(T) = W$. Prove by contradiction, consider two solutions f_1 and f_2:

$$T(f_1) = T(f_2) = g$$

$$0 = T(f_1) - T(f_2) = T(f_1 - f_2)$$

$$\Rightarrow f_1 - f_2 \in ker(T)$$

Since $ker(T) = \{0\}$, $f_1 - f_2 = 0, f_1 = f_2$

3. Span: For any g in W, there exists $T^{-1}(g)$ in V, we can write

$$T^{-1}(g) = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

because f_i span V. Applying T on both sides

$$g = c_1 T(f_1) + c_2 T(f_2) + \cdots + c_n T(f_n)$$

Independence: Consider a relation

$$c_1 T(f_1) + c_2 T(f_2) + \cdots + c_n T(f_n) = 0$$

or

$$T(c_1 f_1 + c_2 f_2 + \cdots + c_n f_n) = 0.$$

Since the ker(T) is $\{0\}$, we have

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0.$$

Since f_i are linear independent, the c_i are all zero.

4. Follows from part (c).

EXAMPLE 6 We are told that the transformation

$$B = T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A - A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

from $R^{2\times 2}$ to $R^{2\times 2}$ is linear. Is T an isomorphism?

Solution We need to examine whether transformation T is invertible. First we try to solve the equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A - A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = B$$

for input A. However, the fact that matrix multiplication is non-commutative gets in the way, and we are unable to solve for A.

Instead, Consider the kernel of T:

$$T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A - A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A = A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We don't really need to find this kernal; we just want to know whether there are nonzero matrices in the kernel. Since I_2 and $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is in the kernel, so that T is not isomophic.

Exercise 4.2: 5, 7, 9, 39