2.3 The Inverse Of a Linear Transformation

Definition. *A function T from X to Y is called* **invertible** *if the equation T(x)=y has a* **unique solution x** *in X for each y in Y.*

Denote the inverse of T as T^{-1} from Y to X, and write

$$T^{-1}(y)=(\text{the unique } x \text{ in } X \text{ such that } T(x) = y)$$

Note

$T^{-1}(T(x)) = x$, for all x in X, and

$T(T^{-1}(y)) = y$, for all y in Y.

If a function T is invertible, then so is T^{-1},

$$(T^{-1})^{-1} = T$$

Consider the case of a *linear transformation* from R^n to R^m given by $\boxed{\vec{y} = A\vec{x}}$ where A is an $m \times n$ matrix, the transformation is invertible if the linear system $\boxed{A\vec{x} = \vec{y}}$ has a unique solution.

1. **Case 1:** $m < n$ The system $A\vec{x} = \vec{y}$ has either no solutions or infinitely many solutions, for any $\vec{y}$ in R^m. Therefore $\vec{y} = A\vec{x}$ is noninvertible.

2. **Case 2:** $m = n$ The system $A\vec{x} = \vec{y}$ has a unique solution iff $rref(A) = I_n$, or equivalently, if $rank(A) = n$.

3. **Case 3:** $m > n$ The transformation $\vec{y} = A\vec{x}$ is noninvertible, because we can find a vector $\vec{y}$ in R^m such that the system $A\vec{x} = \vec{y}$ is inconsistent.

Definition. Invertible Matrix *A matrix A is called invertible if the linear transformation* $\vec{y} = A\vec{x}$ *is invertible. The matrix of inverse transformation is denoted by* A^{-1}. *If the transformation* $\vec{y} = A\vec{x}$ *is invertible. its inverse is* $\vec{x} = A^{-1}\vec{y}$.

Fact

An $m \times n$ matrix A is invertible if and only if

1. A is a square matrix (*i.e.,m*=*n*), and

2. $rref(A) = I_n$.

Example. *Is the matrix A invertible?*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Solution

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{matrix} \longrightarrow \\ -4(I) \\ -7(I) \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \begin{matrix} \longrightarrow \\ \div(-3) \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{bmatrix} \begin{matrix} -2(II) \\ \longrightarrow \\ +6(II) \end{matrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

A fails to be invertible, since $rref(A) \neq I_3$.

Fact Let A be an $n \times n$ matrix.

1. Consider a vector $\vec{b}$ in R^n. If A is invertible, then the system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$. If A is noninvertible, then the system $A\vec{x} = \vec{b}$ has infinitely many solutions or none.

2. Consider the special case when $\vec{b} = \vec{0}$. The system $A\vec{x} = \vec{0}$. has $\vec{x} = \vec{0}$ as a solution. If A is invertible, then this is the only solution. If A is noninvertible, then there are infinitely many other solutions.

If a matrix A is invertible, how can we find the inverse matrix A^{-1}?

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}.$$

or, equivalently, the linear transformation

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 & + & x_2 & + & x_3 \\ 2x_1 & + & 3x_2 & + & 2x_3 \\ 3x_1 & + & 8x_2 & + & 2x_3 \end{bmatrix}.$$

To find the inverse transformation, we solve this system for input variables x_1, x_2, x_3:

$$\left| \begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & = & y_1 & & & & \\ 2x_1 & + & 3x_2 & + & 2x_3 & = & & & y_2 & & \\ 3x_1 & + & 8x_2 & + & 2x_3 & = & & & & & y_3 \end{array} \right| \begin{array}{r} \longrightarrow \\ -2(I) \\ -3(I) \end{array}$$

$$\left| \begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & = & y_1 & & & & \\ & & x_2 & & & = & -2y_1 & + & y_2 & & \\ & & 5x_2 & - & 3x_3 & = & -3y_1 & & & + & y_3 \end{array} \right| \begin{array}{r} -(II) \\ \longrightarrow \\ -5(II) \end{array}$$

$$\left| \begin{array}{rcrcrcrcrcr} x_1 & & & + & x_3 & = & 3y_1 & - & y_2 & & \\ & & x_2 & & & = & -2y_1 & + & y_2 & & \\ & & & - & x_3 & = & 7y_1 & - & 5y_2 & + & y_3 \end{array} \right| \begin{array}{r} \longrightarrow \\ \\ \div(-1) \end{array}$$

$$\left| \begin{array}{rcrcrcrcrcr} x_1 & & & + & x_3 & = & 3y_1 & - & y_2 & & \\ & & x_2 & & & = & -2y_1 & + & y_2 & & \\ & & & & x_3 & = & -7y_1 & + & 5y_2 & - & y_3 \end{array} \right| \begin{array}{r} -(III) \\ \longrightarrow \end{array}$$

$$\left| \begin{array}{rrrcrcrcr} x_1 & & & = & 10y_1 & - & 6y_2 & + & y_3 \\ & x_2 & & = & -2y_1 & + & y_2 & & \\ & & x_3 & = & -7y_1 & + & 5y_2 & - & y_3 \end{array} \right| .$$

We have found the inverse transformation; its matrix is

$$B = A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}.$$

We can write the preceding computations in matrix form:

$$\left[\begin{array}{ccccccc} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 2 & 3 & 2 & : & 0 & 1 & 0 \\ 3 & 8 & 2 & : & 0 & 0 & 1 \end{array}\right] \begin{array}{r} \longrightarrow \\ -2(I) \\ -3(I) \end{array}$$

$$\left[\begin{array}{ccccccc} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & -2 & 1 & 0 \\ 0 & 5 & -1 & : & -3 & 0 & 1 \end{array}\right] \begin{array}{r} -(II) \\ \longrightarrow \\ -5(II) \end{array}$$

$$\left[\begin{array}{ccccccc} 1 & 0 & 1 & : & 3 & -1 & 0 \\ 0 & 1 & 0 & : & -2 & 1 & 0 \\ 0 & 0 & -1 & : & 7 & -5 & 1 \end{array}\right] \begin{array}{r} \longrightarrow \\ \\ \div(-1) \end{array}$$

$$\left[\begin{array}{ccccccc} 1 & 0 & 1 & : & 3 & -1 & 0 \\ 0 & 1 & 0 & : & -2 & 1 & 0 \\ 0 & 0 & 1 & : & -7 & 5 & -1 \end{array}\right] \begin{array}{r} -(III) \\ \longrightarrow \end{array}$$

$$\left[\begin{array}{ccccccc} 1 & 0 & 0 & : & 10 & -6 & 1 \\ 0 & 1 & 0 & : & -2 & 1 & 0 \\ 0 & 0 & 1 & : & -7 & 5 & -1 \end{array}\right].$$

This process can be described succinctly as follows:

Find the inverse of a matrix

To find the inverse of an $n \times n$ matrix A, from the $n \times (2n)$ matrix $\left[\begin{array}{c:c} A & I_n \end{array}\right]$ and compute rref $\left[\begin{array}{c:c} A & I_n \end{array}\right]$.

- If rref $[A \vdots I_n]$ is of the form $[I_n \vdots B]$, then A is invertible, and $A^{-1} = B$.

- If rref $[A \vdots I_n]$ is of another form (*i.e., its left half fails to be* I_n), then A is not invertible. (*Note that the left half of rref* $[A \vdots I_n]$ *is rref(A).*)

The inverse of a 2×2 matrix is particularly easy to find.

Inverse and determinant of a 2×2 matrix

1. The 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

   is invertible if (and only if) $ad - bc \neq 0$. Quantity $\boxed{ad - bc}$ is called the determinant of A, written det(A):

$$det(A) = det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

2. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\mathrm{ad-bc}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\mathrm{det(A)}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

   Compare this with Exercise 2.1.13.

Homework. *Exercise 2.3 21–27, 41*