2.2 Linear Transformation in Geometry

Example. 1 *Consider a linear transformation system* $T(\vec{x}) = A\vec{x}$ *from* R^n *to* R^m*.*

a. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$

In words, the transformation of the sum of two vectors equals the sum of the transformation.

b. $T(k\vec{v}) = kT(\vec{v})$

In words, the transformation of a scalar multiple of a vector is the scalar multiple of the transform.

See Figure 1 (pp.50).

Fact A transformation T from R^n to R^m is linear iff

a. $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$, for all $\vec{v}$, $\vec{w}$ in R^n, and

b. $T(k\vec{v}) = kT(\vec{v})$, for all $\vec{v}$ in R^n and all scalars k.

Proof

Idea: To prove the inverse, we must show a matrix A such that $T(\vec{x}) = A\vec{x}$. Consider a transformation T from R^n to R^m that satisfy (a) and (b), find A.

Example. 2 *Consider a linear transformation T from R^2 to R^2. The vectors $T\vec{e}_1$ and $T\vec{e}_2$ are sketched in Figure 2. Sketch the* **image** *of the unit square under this transformation.*

See Figure 2. (pp. 51)

Example. 3 *Consider a linear transformation T from R^2 to R^2 such that $T(\vec{v}_1) = \frac{1}{2}\vec{v}_1$ and $T(\vec{v}_2) = 2\vec{v}_2$, for the vectors $\vec{v}_1$ and $\vec{v}_2$ in Figure 5. On the same axes, sketch $T(\vec{x})$, for the given vector $\vec{x}$.*

See Figure 5. (pp. 52)

[Rotation]

Example. 4 *Let T be the counterclockwise rotation through an angle α.*

a. Draw sketches to illustrate that T is a linear transformation.

b. Find the matrix of T.

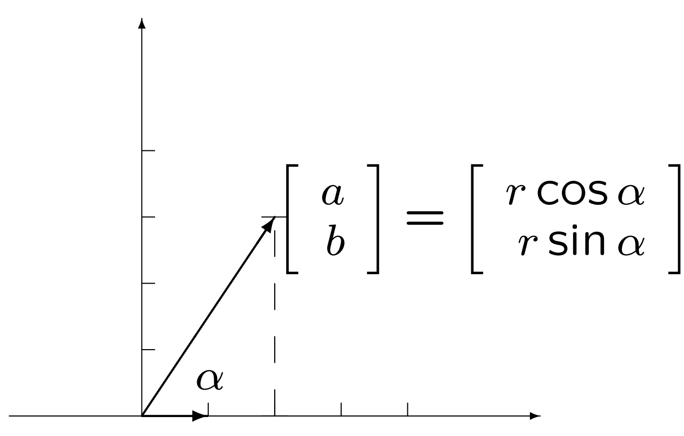
Example. 5 *Give a geometric interpretation of the linear transformation.*

$$T(\vec{x}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{x}$$

Rotation-dilations A matrix with this form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

denotes a counterclockwise rotation through the angle α followed by a dilation by the factor r where $\tan(\alpha) = \frac{b}{a}$ and $r = \sqrt{a^2 + b^2}$. Geometrically,



[Shears]

Example. 6 *Consider the linear transformation*

$$T(\vec{x}) = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \vec{x}$$

To understand this transformation, sketch the image of the **unit square***.*

Solution The transformation $T(\vec{x}) = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \vec{x}$ is called a *shear* parallel to the x_1-axis.

Definition. Shear *Let L be a line in R^2. A linear transformation T from R^2 to R^2 is called a* **shear parallel** *to L if*

a. $T(\vec{v}) = \vec{v}$, for all vectors $\vec{v}$ on L, and

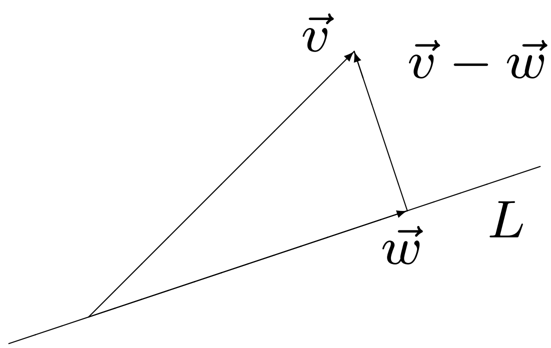
b. $T(\vec{v}) - \vec{v}$ is parallel to L for all vectors $\vec{x} \in R^2$.

Example. 7 *Consider two perpendicular vectors $\vec{u}$ and $\vec{w}$ in R^2. Show that the transformation*

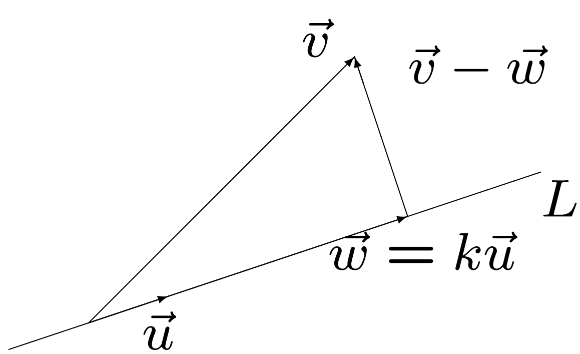
$$T(\vec{x}) = \vec{x} + (\vec{u} \cdot \bar{x})\vec{w}$$

is a shear parallel to the line L spanned by $\vec{w}$.

Consider a line L in R^2. For any vector $\vec{v}$ in R^2, there is a unique vector $\vec{w}$ on L such that $\vec{v} - \vec{w}$ is perpendicular to L.



How can we generalize the idea of an orthogonal projection to lines in R^n?



Definition. orthogonal projection *Let L be a line in R^n consisting of all scalar multiples of some unit vector $\vec{u}$. For any vector $\vec{v}$ in R^n there is a unique vector $\vec{w}$ on L such that $\vec{v}-\vec{w}$ is perpendicular to L, namely, $\vec{w} = (\vec{u}\cdot\vec{v})\vec{u}$. This vector $\vec{w}$ is called the* **orthogonal projection** *of $\vec{v}$ onto L:*

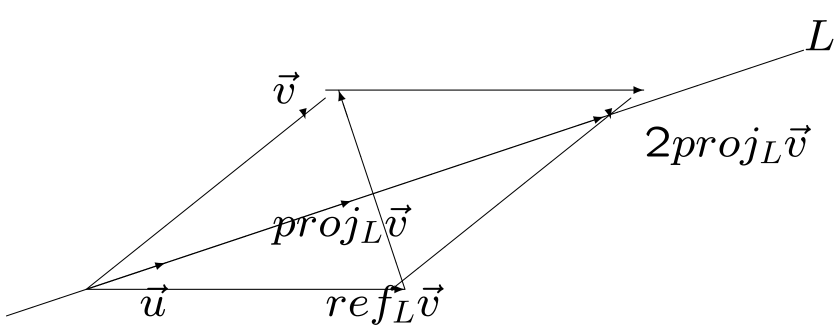
$$proj_L(\vec{v}) = (\vec{u} \cdot \vec{v})\vec{u}$$

The transformation $proj_L$ from R^n to R^n is linear.

Definition. *Let L be a line in R^n, the vector $2(proj_L\vec{v}) - \vec{v}$ is called the* **reflection** *of $\vec{v}$ in L:*

$$ref_L(\vec{v}) = 2(proj_L\vec{v}) - \vec{v} = 2(\vec{u} \cdot \vec{v})\vec{u} - \vec{v}$$

where $\vec{u}$ is a unit vector on L.



Homework. *Exercise 2.2: 1, 9, 13, 17, 27*