Applied Linear Algebra

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Chapter 3
Subspaces of R^n and Their Dimensions

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3.1 Image and Kernal of a Linear Transformation

Definition. Image
The image of a function consists of all the values the function takes in its codomain. If f is a function from X to Y, then

$$\begin{aligned} \text{image(f)} &= \{f(x)\text{: } x \in X\} \\ &= \{y \in Y\text{: } y = f(x), \text{ for some } x \in X\} \end{aligned}$$

Example. *See Figure 1.*

Example. *The image of*

$$f(x) = e^x$$

consists of all positive numbers.

Example. $b \in im(f), c \notin im(f)$ *See Figure 2.*

Example. $f(t) = \begin{bmatrix} cos(t) \\ sin(t) \end{bmatrix}$ *(See Figure 3.)*

Example. *If the function from X to Y is invertible, then $image(f) = Y$. For each y in Y, there is one (and only one) x in X such that $y = f(x)$, namely, $x = f^{-1}(y)$.*

Example. *Consider the linear transformation T from R^3 to R^3 that projects a vector orthogonally into the $x_1 - x_2$-plane, as illustrate in Figure 4. The image of T is the $x_1 - x_2$-plane in R^3.*

Example. *Describe the image of the linear transformation T from R^2 to R^2 given by the matrix*

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

Solution

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= (x_1 + 3x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

See Figure 5.

Example. *Describe the image of the linear transformation T from R^2 to R^3 given by the matrix*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

See Figure 6.

Definition. *Consider the vectors $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_n$ in R^m. The set of all linear combinations of the vectors $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_n$ is called their* **span***:*

span$(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$
$=\{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$: c_i *arbitrary scalars*$\}$

Fact The image of a linear transformation

$$T(\vec{x}) = A\vec{x}$$

is the span of the columns of A. We denote the image of T by $im(T)$ or $im(A)$.

Justification

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v_1} & \ldots & \vec{v_n} \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\vec{v_1} + x_2\vec{v_2} + \ldots + x_n\vec{v_n}.$$

Fact: Properties of the image

(a). The zero vector is contained in $im(T)$, i.e. $\vec{0} \in im(T)$.

(b). The image is closed under addition: If $\vec{v}_1$, $\vec{v}_2 \in im(T)$, then $\vec{v}_1 + \vec{v}_2 \in im(T)$.

(c). The image is closed under scalar multiplication: If $\vec{v} \in im(T)$, then $k\vec{v} \in im(T)$.

Verification

(a). $\vec{0} \in R^m$ since $A\vec{0} = \vec{0}$.

(b). Since $\vec{v_1}$ and $\vec{v_2} \in im(T)$, $\exists$ $\vec{w_1}$ and $\vec{w_2}$ st. $T(\vec{w_1}) = \vec{v_1}$ and $T(\vec{w_2}) = \vec{v_2}$. Then, $\vec{v_1} + \vec{v_2} = T(\vec{w_1}) + T(\vec{w_2}) = T(\vec{w_1} + \vec{w_2})$, so that $\vec{v_1} + \vec{v_2}$ is in the image as well.

(c). $\exists$ $\vec{w}$ st. $T(\vec{w}) = \vec{v}$. Then $k\vec{v} = kT(\vec{w}) = T(k\vec{w})$, so $k\vec{v}$ is in the image.

Example. *Consider an* $n \times n$ *matrix* A. *Show that* $im(A^2)$ *is contained in* $im(A)$.

Hint: To show $\vec{w}$ *is also in* $im(A)$, *we need to find some vector* $\vec{u}$ *st.* $\vec{w} = A\vec{u}$.

Solution

Consider a vector $\vec{w}$ *in* $im(A^2)$. *There exists a vector* $\vec{v}$ *st.* $\vec{w} = A^2\vec{v} = AA\vec{v} = A\vec{u}$ *where* $\vec{u} = A\vec{v}$.

Definition. *Kernel*

The kernel of a linear transformation $T(\vec{x}) = A\vec{x}$ is the set of all zeros of the transformation (i.e., the solutions of the equation $A\vec{x} = \vec{0}$. See Figure 9.

We denote the kernel of T by $ker(T)$ or $ker(A)$.

For a linear transformation T from R^n to R^m,

- $im(T)$ is a subset of the codomain R^m of T, and

- $ker(T)$ is a subset of the domain R^n of T.

Example. *Consider the orthogonal project onto the* $x_1 - x_2-$*plane, a linear transformation* T *from* R^3 *to* R^3*. See Figure 10.*

The kernel of T *consists of all vectors whose orthogonal projection is* $\vec{0}$*. These are the vectors on the* x_3-*axis (the scalar multiples of* $\vec{e}_3$*).*

Example. *Find the kernel of the linear transformation T from R^3 to R^2 given by*

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution

We have to solve the linear system

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x} = \vec{0}$$

$$rref \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\left| \begin{array}{ccccc} x_1 & & - & x_3 & = 0 \\ & x_2 & + & 2x_3 & = 0 \end{array} \right|$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel is the line spanned by $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Example. *Find the kernel of the linear transformation T from R^5 to R^4 given by the matrix*

$$A = \begin{bmatrix} 1 & 5 & 4 & 3 & 2 \\ 1 & 6 & 6 & 6 & 6 \\ 1 & 7 & 8 & 10 & 12 \\ 1 & 6 & 6 & 7 & 8 \end{bmatrix}$$

Solution We have to solve the linear system $\mathrm{T}(\vec{x}) = \mathrm{A}\vec{0} = \vec{0}$

$$\mathrm{rref(A)} = \begin{bmatrix} 1 & 0 & -6 & 0 & 6 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The kernel of T consists of the solutions of the system

$$\left| \begin{array}{lllll} x_1 & & -6x_3 & & +6x_5 = 0 \\ & x_2 & +2x_3 & & -2x_5 = 0 \\ & & & x_4 & +2x_5 = 0 \end{array} \right|$$

The solution are the vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6s - 6t \\ -2s + 2t \\ s \\ -2t \\ t \end{bmatrix}$$

where s and t are arbitrary constants .

$$\text{ker(T)=} \begin{bmatrix} 6s - 6t \\ -2s + 2t \\ s \\ -2t \\ t \end{bmatrix} : \text{ s , t arbitrary scalars}$$

We can write

$$\begin{bmatrix} 6s - 6t \\ -2s + 2t \\ s \\ -2t \\ t \end{bmatrix} = \text{s} \begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \text{t} \begin{bmatrix} -6 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

This shows that

$$\ker(T) = \operatorname{span}\left(\begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right)$$

Fact 3.1.6: Properties of the kernel

(a) The zero vector $\vec{0}$ in R_n in in ker(T).
(b) The kernel is closed under addition.
(c) The kernel is closed under scalar multiplication.

The verification is left as Exercise 49.

Fact 3.1.7
1. Consider an m*n matrix A then

$$\text{ker(A)} = \{\vec{0}\}$$

if (and only if) rank(A) $= n$.(This implies that $n \leq m$.)

Check exercise 2.4 (35)

2. For a square matrix A,

$$\text{ker(A)} = \{\vec{0}\}$$

if (and only if) A is invertible.

Summary

Let A be an n*n matrix . The following statements are equivalent (i.e.,they are either all true or all false):

1. A is invertible.

2. The linear system A$\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, for all $\vec{b}$ in R^n. (def 2.3.1)

3. rref(A) $= I_n$. (fact 2.3.3)

4. rank(A) $= n$. (def 1.3.2)

5. im(A) $= R^n$. (ex 3.1.3b)

6. ker(A) $= \{\vec{0}\}$. (fact 3.1.7)

Homework 3.1: *5, 6, 7, 14, 15, 16, 31, 33, 42, 43*

3.2 Subspaces of R^n Bases and Linear Independence

Definition. Subspaces of R^n

A subset W of R^n is called a subspace of R^n if it has the following properties:

(a). W contains the zero vector in R^n.
(b). W is closed under addition.
(c). W is closed under scalar multiplication.

Fact 3.2.2

If T is a linear transformation from R^n to R^m, then

$\diamond$ $ker(T)$ is a subspace of R^n
$\diamond$ $im(T)$ is a subspace of R^m

Example. *Is* $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 : x \geq 0, y \geq 0 \right\}$ *a subspace of* R^2*?*

See Figure 1, 2.

Example. *Is* $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 : xy \geq 0 \right\}$ *a subspace of* R^2*?*

See Figure 3, 4.

Example. *Show that the only subspaces of* R^2 *are:* $\{\vec{0}\}$*, any lines through the origin, and* R^2 *itself.*

Similarly, the only subspaces of R^3 are: $\{\vec{0}\}$, any lines through the origin, any planes through $\vec{0}$, and R^3 itself.

Solution
Suppose W is a subspace of R^2 that is neither the set $\{\vec{0}\}$ nor a line through the origin. We have to show $W = R^2$.

Pick a nonzero vector $\vec{v_1}$ in W. (We can find such a vector, since W is not $\{\vec{0}\}$.) The subspace W contains the line L spanned by $\vec{v_1}$, but W does not equal L. Therefore, we can find a vector $\vec{v_2}$ in W that is not on L (See Figure 5). Using a parallelogram, we can express any vector $\vec{v}$ in R^2 as a linear combination of $\vec{v_1}$ and $\vec{v_2}$. Therefore, $\vec{v}$ is contained in W (Since W is closed under linear combinations). This shows that $W = R^2$, as claimed.

A plane E in R^3 is usually described either by

$$x_1 + 2x_2 + 3x_3 = 0$$

or by giving E parametrically, as the span of two vectors, for example,

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

In other words, E is described either as

$$ker \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

or

$$im \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & 1 \end{bmatrix}$$

Similarly, a line L in R^3 may be described either parametrically, as the span of the vector

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

or by two linear equations

$$\left| \begin{array}{c} x_1 - x_2 - x_3 = 0 \\ x_1 - 2x_2 + x_3 = 0 \end{array} \right|$$

Therfore

$$L = im \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = ker \begin{bmatrix} 1 & -1 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

A subspace of R^n is uaually presented either as the solution set of a homogeneous linear system (as a kernel) or as the span of some vectors (as an image).

Any subspace of R^n can be represented as the image of a matrix.

Bases and Linear Independence

Example. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 2 & 4 \end{bmatrix}$$

Find vectors $\vec{v_1}$, $\vec{v_2}$, $\cdots$, $\vec{v_m}$ *in* R^3 *that span the image of* A. *What is the* <u>*smallest number*</u> *of vectors needed to span the image of* A*?*

Solution

We know from Fact 3.1.3 that the image of A spanned by the columns of A,

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \vec{v_4} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Figure 6 show that we need only $\vec{v_1}$ and $\vec{v_2}$ to span the image of A. Since $\vec{v_3} = \vec{v_2}$ and $\vec{v_4} = \vec{v_1} + \vec{v_2}$, the vectors $\vec{v_3}$ and $\vec{v_4}$ are redundant; that is, they are linear combinations of $\vec{v_1}$ and $\vec{v_2}$:

$$\begin{aligned} \text{im(A)} &= \text{span}(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}) \\ &= \text{span}(\vec{v_1}, \vec{v_2}) \ . \end{aligned}$$

The image of A can be spanned by two vectors, but not by one vectors alone.

Definition. Linear independence; basis
Consider a sequence $\vec{v}_1, \ldots, \vec{v}_m$ *of vectors in a subspace* V *of* R^n*.*

The vectors $\vec{v}_1, \ldots, \vec{v}_m$ are called **linearly independent** if nono of them is a linear combination of the others.

We say that the vectors $\vec{v}_1, \ldots, \vec{v}_m$ form a **basis** of V if they span V and are linearly independent.

See last example. The vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$, $\vec{v_4}$ span

$$V = im(A)$$

but they are linearly dependent, because $\vec{v_4}=\vec{v_2}+\vec{v_3}$. Therefore, they do not form a basis of V. The vectors $\vec{v_1}$, $\vec{v_2}$, on the other hand, do span V and are linearly independent.

Definition. Linear relations

Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m$ *in* R^n*. An equation of the form*

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m = \vec{0}$$

is called a **(linear) relation** *among the vectors* $\vec{v}_i$*. There is always the trievial relation, with* $c_1 = c_2 = \cdots = c_m = 0$*. Nontrivial relations may or may not exist among the vectors* $\vec{v}_1, \ldots, \vec{v}_m$ *in* R^n*.*

Fact 3.2.5

The vectors $\vec{v}_1, \ldots, \vec{v}_m$ in R^n are linearly dependent if (and only if) there are nontrivial relations among them.

Proof

$\Rightarrow$ If one of the $\vec{v}_i$ s a linear combination of the others,

$$\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \ldots + c_m\vec{v}_m$$

then we can find a nontrivial relation by subtracting $\vec{v}_i$ from both sides of the equations:

$$c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} - \vec{v}_i + c_{i+1}\vec{v}_{i+1} + \ldots + c_m\vec{v}_m = \vec{0}$$

$\Leftarrow$ Conversely, if there is a nontrivial relation

$$c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \ldots + c_m\vec{v}_m = \vec{0}$$

then we can solve for $\vec{v}_i$ and express $\vec{v}_i$ as a linear combination of the other vectors.

Example. *Determine whether the following vectors are linearly independent*

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{bmatrix}.$$

Solution

TO find the relations among these vectors, we have to solve the vector equation

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \\ 11 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 6 & 2 & 1 \\ 2 & 7 & 3 & 4 \\ 3 & 8 & 5 & 9 \\ 4 & 9 & 7 & 16 \\ 5 & 10 & 11 & 25 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In other words, we have to find the $kernal$ of A. To do so, we compute $rref(A)$. Using technology, we find that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows the kernel of A is $\{\vec{0}\}$, because there is a leading 1 in each column of $rref(A)$. There is only the trivial relation among the four vectors and they are therefore linearly independent.

Fact 3.2.6

The vectors $\vec{v}_1, \ldots, \vec{v}_m$ in R^n are linearly independent if (and only if)

$$ker \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_m \\ | & | & & | \end{bmatrix} = \{\vec{0}\}$$

or, equivalently, of

$$rank \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_m \\ | & | & & | \end{bmatrix} = m$$

This condition implies that $m \leq n$.

Fact 3.2.7
Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in a subspace V of R^n.
The vectors $\vec{v}_i$ are a basis of V if (and only if) every vector $\vec{v}$ in V can be expressed **uniquely** as a linear combination of the vectors $\vec{v}_i$.

Proof
$\Rightarrow$ Suppose vectors $\vec{v}_i$ are a basis of V, and consider a vector $\vec{v}$ in V. Since the basis vectors span V, the vector $\vec{v}$ can be written as a linear combination of the $\vec{v}_i$. We have to demonstrate that this representation is unique. If there are two representations:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m$$
$$= d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_m\vec{v}_m$$

By subtraction, we find

$$\vec{0} = (c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \ldots + (c_m - d_m)\vec{v}_m$$

Since the $\vec{v}_i$ are linearly independent, $c_i - d_i = 0$, or $c_i = d_i$, for all i.

$\Leftarrow$, suppose that each vector in V can be expressed uniquely as a linear combination of the vectors $\vec{v}_i$. Clearly, the $\vec{v}_i$. span V. The zero vector can be expressed uniquely as a linear combination of the $\vec{v}_i$, namely, as

$$\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \ldots + 0\vec{v}_m$$

This means there is only the trivial relation among the $\vec{v}_i$: they are linearly independent.

See Figure 7. The vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$, $\vec{v_4}$ do not form a basis of E, since every vector in E can be expressed in more than one way as a linear combination of the $\vec{v}_i$. For example,

$$\vec{v}_4 = \vec{v}_1 + \vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4$$

but also

$$\vec{v}_4 = 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + 1\vec{v}_4.$$

Homework 3.2: *3, 5, 9, 17, 18, 19, 29, 30, 39*

3.3 The Dimension of a Subspace of R^n

Fact 3.3.2
All bases of a subspace V of R^n consist of the same number of vectors.

Hint Basis: linear independent and span V (Def 3.2.3)

Fact 3.3.1
Consider vectors $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_p}$ and $\vec{w_1}$, $\vec{w_2}$, ..., $\vec{w_q}$ in a subspace V of R^n. If the vectors $\vec{v_i}$ are linearly independent, and the vectors $\vec{w_j}$ span V, then p $\leq$ q.

Proof 3.3.2

Consider two bases $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_p}$ and $\vec{w_1}$, $\vec{w_2}$, ..., $\vec{w_q}$ of V. Since the $\vec{v_i}$ are linearly independent, and the vectors $\vec{w_j}$ span V, we have $p \leq q$. Like wise, since the $\vec{w_j}$ are linearly independent and the $\vec{v_i}$ span V, we have $q \leq p$. Therefore, $p = q$.

Proof 3.3.1

$$\begin{array}{ccccc} \vec{v}_1 & = & a_{11}\vec{w}_1 & +\cdots+ & a_{1q}\vec{w}_q \\ \vdots & & \vdots & & \vdots \\ \vec{v}_p & = & a_{p1}\vec{w}_1 & +\cdots+ & a_{pq}\vec{w}_q \end{array}$$

Write each of these equations in matrix form:

$$\begin{bmatrix} | & & | \\ \vec{w}_1 & \dots & \vec{w}_q \\ | & & | \end{bmatrix} \begin{bmatrix} a_{11} \\ \vdots \\ a_{1q} \end{bmatrix} = \vec{v}_1$$

$$\dots$$

$$\begin{bmatrix} | & & | \\ \vec{w}_1 & \dots & \vec{w}_q \\ | & & | \end{bmatrix} \begin{bmatrix} a_{p1} \\ \vdots \\ a_{pq} \end{bmatrix} = \vec{v}_p$$

Combine all these equations into one matrix equation:

$$\begin{bmatrix} | & & | \\ \vec{w}_1 & \dots & \vec{w}_q \\ | & & | \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{p1} \\ \vdots & & \vdots \\ a_{1q} & \dots & a_{pq} \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_p \\ | & & | \end{bmatrix}$$

$$MA = N$$

Because

$$A\vec{x} = \vec{0}, MA\vec{x} = N\vec{x} = \vec{0}$$

The kernel of A is contained in the kernel of N.

Since the kernel of N is $\{\vec{0}\}$ (since the $\vec{v}_i$ are linearly independent), the kernel of A is $\{\vec{0}\}$ as well.

This implies that $rank(A) = p \leq q$ (by Fact 3.1.7).

Definition. Dimension
Consider a subspace V of R^n. The number of vectors in a basis of V is called the **dimension** *of V, denoted by $dim(V)$.*

What is the dimension R^n itself?

Clearly, R^n ought to have dimension n. This is indeed the case: the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ form a basis of R^n called its **standard basis**.

A plane E in R^3 is two-dimensional.

Fact 3.3.4

Consider a subspace V of R^n with $dim(V) = m$

1. We can find at most m linearly independent vectors in V.

2. We need at least m vectors to span V.

3. If m vectors in V are linearly independent, then they form a basis of V.

4. If m vectors span V, then they form a basis of V.

Proof 3.3.4 (3)

Consider linearly independent vectors $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_m}$ in V. We have to show that the $\vec{v}_i$ span V. Pick a $\vec{v}$ in V. Then the vectors $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_m}, \vec{v}$ will be linearly dependent, by (1). Therefore, there is a nontrivial relation

$$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m + c\vec{v} = \vec{0}$$

We can solve the relation for $\vec{v}$ and express it as a linear combination of the $\vec{v}_i$. In other words, the $\vec{v}_i$ span V.

Finding a Basis of the Kernel

Example. *Find a basis of the kernel of the following matrix, and determine the dimension of the kernel:*

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 2 & 4 & 1 & 9 & 5 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 2 & 4 & 1 & 9 & 5 \end{bmatrix} \begin{matrix} \\ -2(I) \end{matrix}$$

$$\longrightarrow rref(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}$$

This corresponds to the system

$$\left| \begin{array}{lllll} x_1 + 2x_2 & & 3x_4 & & = 0 \\ & x_3 + & 3x_4 & +5x_5 & = 0 \end{array} \right|$$

with general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s-3t \\ s \\ -3t-5r \\ t \\ r \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad\quad \uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$
$$\qquad\qquad\qquad\qquad\quad \vec{v}_1 \qquad\quad\; \vec{v}_2 \qquad\quad\; \vec{v}_3$$

The tree vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ span $ker(A)$ and form a basis of the kernel of A (i.e. linearly independent).

dim(ker A)=(number of nonleading variables)
=(number of columns of A)-(number of leading variables)
=(number of columns of A)-rank(A)
=5-2 =3

Fact 3.3.5
Consider an $m \times n$ matrix A.

$$dim(kerA) = n - rank(A)$$

Finding a Basis of the Image

Example. *Find a basis of the image of the linear transformation* T *from* R^5 *to* R^4 *with matrix*

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}$$

and determine the dimenson of the image.

Solution

We know the columns of A span the image of A, but they are linearly dependent in this example. To construct a basis of im(A), we could find a relation among the columns of A, express one of the columns as linear combinartion of the others, and then omit this vector as redundant.

We first find the reduced row-echelon form of A:

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{matrix}$$

$$E = rref(A) = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 \end{matrix}$$

By inspection, we can express any column of $rref(A)$ that does not contain a leading 1 as a linear combination of earlier columns that do contain a leading 1.

$$\vec{w}_3 = \vec{w}_1 - 2\vec{w}_2, \text{ and } \vec{w}_4 = 2\vec{w}_1 - 3\vec{w}_2$$

It may surprise you that the same relationships hold among the corresponding columns of the matrix A.

$$\vec{v}_3 = \vec{v}_1 - 2\vec{v}_2, \text{ and } \vec{v}_4 = 2\vec{v}_1 - 3\vec{v}_2$$

Since $\vec{w_1}$, $\vec{w_2}$, and $\vec{w_5}$ are linearly independent, so are the vectors $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_5}$. (Why?)

The vectors $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_5}$ alone span the image of A, since any vector $\vec{v}$ in the image of A can be expressed as

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4} + c_5\vec{v_5}$$

$$= c_1\vec{v_1} + c_2\vec{v_2} + c_3(\vec{v_1} - 2\vec{v_2}) + c_4(2\vec{v_1} - 3\vec{v_2}) + c_5\vec{v_5}$$

Therefore, the vectors $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_5}$ form a basis of $im(A)$, and thus $dim(imA) = 3$.

Definition.

A column of a matrix A is called a **pivot column** if the corresponding column of $rref(A)$ contains a leading 1.

Fact 3.3.7 The pivot columns of a matrix A form a basis of $im(A)$.

Fact 3.3.8 For any matrix A,

$$rank(A) = dim(imA).$$

Fact 3.3.9 Rank-Nullity Theorem
If A is an $m \times n$ matrix, then

$$dim(ker A) + dim(im A) = n.$$

The dimension of the kernel of matrix A is called the **nullity** of A:

$$nullity(A) = dim(ker A).$$

Using this definition and Fact 3.3.8, we can write:

$$nullity(A) + rank(A) = n.$$

$\Rightarrow$ The larger the kernel, the smaller the image, and vice versa.

Bases of R^n

How can we tell n given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in R^n form a basis?

The $\vec{v}_i$ form a basis of R^n if every vector $\vec{b}$ in R^n can be written uniquely as a linear combination of the $\vec{v}_i$:

$$\vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The linear system

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{b}$$

has a unique solution if (only if) the $n \times n$ matrix

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

is invertible.

Fact 3.3.10 The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in R^n form a basis of R^n if (and only if) the matrix

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

is invertible.

Example. *Are the following vectors a basis of R^4?*

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 9 \\ 1 \end{bmatrix}, \ \vec{v_2} = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 8 \end{bmatrix}, \ \vec{v_3} = \begin{bmatrix} 1 \\ 8 \\ 1 \\ 5 \end{bmatrix}, \ \vec{v_4} = \begin{bmatrix} 1 \\ 9 \\ 7 \\ 3 \end{bmatrix}$$

Solution

We have to check whether the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 9 \\ 9 & 4 & 1 & 7 \\ 1 & 8 & 5 & 3 \end{bmatrix}$$

is invertible. Using technology, we find that

$$\text{reff} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 9 \\ 9 & 4 & 1 & 7 \\ 1 & 8 & 5 & 3 \end{bmatrix} = I_4$$

Thus, the vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$, $\vec{v_4}$ form a basis of R^4

Summary 3.3.11

Consider an $n \times n$ matrix

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

Then the following statements are equivalent:

1. A is invertible.

2. The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, for all $\vec{b}$ for all $\vec{b}$ in R^n.

3. $rref(A) = I_n$.

4. $rank(A) = n$.

5. $im(A) = R^n$.

6. $ker(A) = \{\vec{0}\}$.

7. The $\vec{v}_i$ are a basis of R^n.

8. The $\vec{v}_i$ span R^n.

9. The $\vec{v}_i$ are linearly independent.

Homework 3.3 *6, 7, 8, 17, 18, 27, 31, 33, 39, 58, 59*

Exercise 49: Find a basis of the row space of the matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 51: Consider an arbitrary $m \times n$ matrix A.

1. What is the relationship between the row spaces of A and $E = rref(A)$?

2. What is the relationship between the dimension of the row space of A and the rank of A?

3.4 COORDINATES

EXAMPLE 1

Let V be the plane in R^3 with equation $x_1+2x_2+3x_3=0$, a two-dimensional subspace of R^3. We can describe a vector in this plane by its spatial (3D)coordinates; for example, vector

$$\vec{x} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$$

is in plane V. However, it may be more convenient to introduce a plane coordinate system in V.

Consider any two vectors in plane V that aren't parallel, e.g.

$$\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \text{ and } \vec{v_2} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

See Figure 1, where we label the new axes c_1 and c_2, with the new coordinate grid defined by vectors $\vec{v_1}$ and $\vec{v_2}$.

Note that the c_1 - c_2 coordinates of vector $\vec{v_1}$ is $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the coordinates of vector $\vec{v_2}$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively.

For a vector $\vec{x}$ in plane V, we can find the scalars c_1 and c_2 such that

$$\vec{x} = c_1\vec{v_1} + c_2\vec{v_2}.$$

For example, $\vec{x} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Therefore, the $c_1 - c_2$ coordinates of $\vec{x}$ are

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

See Figure 3.

Let's denote the basis $\vec{v_1}$, $\vec{v_2}$ of V by B (Fraktur B). Then, the coordinate vector of $\vec{x}$ with respect to B is denoted by $\begin{bmatrix} \vec{x} \end{bmatrix}_B$:

$$\text{If } \vec{x} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}, \text{ then } \begin{bmatrix} \vec{x} \end{bmatrix}_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Definition 3.4.1
Coordinates in a subspace of R^n
Consider a basis B of a subspace V of R^n, consisting of vectors $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_m}$. Any vector $\vec{x}$ in V can be written uniquely as

$$\vec{x}=c_1\vec{v_1}+c_2\vec{v_2}+...+c_m\vec{v_m}$$

The scalars c_1, c_1, ..., c_m are called the B-coordinates of $\vec{x}$, and the vector

$$\begin{bmatrix} c_1 \\ c_2 \\ ... \\ c_m \end{bmatrix}$$

is called the B-coordinate vector of $\vec{x}$, denoted by $\left[\ \vec{x}\ \right]_B$.

Note that

$$\vec{x} = S\left[\ \vec{x}\ \right]_B$$

where $S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & & | \end{bmatrix}$, an n$\times m$ matrix.

EXAMPLE 2

Consider the basis B of R^2 consisting of vectors $\vec{v_1}= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v_2}= \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

a. If $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, find $\left[\vec{x} \right]_B$

b. If $\left[\vec{x} \right]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $\vec{x}$

Solution

a. To find the coordinates of vector $\vec{x}$, we need to write $\vec{x}$ as a linear combination of the basis vectors:

$\vec{x} = c_1\vec{v_1} + c_2\vec{v_2}$, or $\begin{bmatrix} 10 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Alternatively, we can solve the equation

$$\vec{x} = S \left[\vec{x} \right]_B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \left[\vec{x} \right]_B$$

for $\left[\vec{x} \right]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\left[\ \vec{x}\ \right]_B = S^{-1}\vec{x} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

b. By definition of coordinates, $\left[\ \vec{x}\ \right]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ means that

$$\vec{x} = 2\vec{v_1} + (-1)\vec{v_2} = 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

Alternatively, use the formula

$$\vec{x} = S\left[\ \vec{x}\ \right]_B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

EXAMPLE 3

Let L be the line in R^2 spanned by vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Let T be the linear transformation from R^2 to R^2 that projects any vector orthogonally onto line L, as shown in Figure 5.

1. In $\vec{x}_1 - \vec{x}_2$ coordinate system (See Figure 5): Sec 2.2 (pp. 59).

2. In $c_1 - c_2$ coordinate system (See Figure 6): T transforms vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ into $\begin{bmatrix} c_1 \\ 0 \end{bmatrix}$.

   That is, T is given by the matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$

The transforms from $\left[\ \vec{x}\ \right]_B$ into $\left[\ T(\vec{x})\ \right]_B$ is called the B-matrix of T:

$$\left[\ T(\vec{x})\ \right]_B = B \left[\ \vec{x}\ \right]_B$$

Definition 3.4.2
The B-matrix of a linear transformation
Consider a linear transformation T from R^n to R^n and a basis B of R^n. The $n \times n$ matrix B that transforms $\left[\vec{x} \right]_B$ into $\left[T(\vec{x}) \right]_B$ is called the B-matrix of T:

$$\left[T(\vec{x}) \right]_B = \mathrm{B} \left[\vec{x} \right]_B$$

for all $\vec{x}$ in R^n.

Fact 3.4.3 The columns of the B-matrix of a linear transformation
Consider a linear transformation T from R^n to R^n and a basis B of R^n consisting of vectors $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_n}$. Then, the B-matrix of T is

$$B = \left[\left[T(\vec{x_1}) \right]_B \left[T(\vec{x_2}) \right]_B \cdots \left[T(\vec{x_n}) \right]_B \right]$$

That is, the columns of B are the B-coordinate vectors of T($\vec{v_1}$), T($\vec{v_2}$), ..., T($\vec{v_n}$).

EXAMPLE 4

Consider two perpendicular unit vectors $\vec{v_1}$ and $\vec{v_2}$ in R^3. Form the basis $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$=$\vec{v_1} \times \vec{v_2}$ of R^3; let's denote this basis by B. Find the B-matrix B of the linear transformation T($\vec{x}$)=$\vec{v_1} \times \vec{x}$.

(see Exercise 2.1: 44 on pp. 49,
$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix})$$

Solution

Use Fact 3.4.3 to construct B column by column:

$$B = \begin{bmatrix} \begin{bmatrix} T(\vec{x_1}) \end{bmatrix}_B \begin{bmatrix} T(\vec{x_2}) \end{bmatrix}_B \cdots \begin{bmatrix} T(\vec{x_n}) \end{bmatrix}_B \end{bmatrix}$$
$$= \begin{bmatrix} \begin{bmatrix} \vec{v_1} \times \vec{v_1} \end{bmatrix}_B \begin{bmatrix} \vec{v_1} \times \vec{v_2} \end{bmatrix}_B \begin{bmatrix} \vec{v_1} \times \vec{v_3} \end{bmatrix}_B \end{bmatrix}$$
$$= \begin{bmatrix} \begin{bmatrix} \vec{0} \end{bmatrix}_B \begin{bmatrix} \vec{v_3} \end{bmatrix}_B \begin{bmatrix} -\vec{v_2} \end{bmatrix}_B \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

EXAMPLE 5

Let T be the linear transformation from R^2 to R^2 that projects any vector orthogonally onto the line L spanned by $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. In Example 3, we found that the matrix of T with respect to the basis B consisting of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

What is the relation ship between B and the standard matrix A of T (such that T($\vec{x}$)=A$\vec{x}$)?

Solution

Recall from Definition 3.4.1 that

$$\vec{x} = S \left[\ \vec{x} \ \right]_B, \text{ where } S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

and consider the following diagram: (Figure 7)

Note that T($\vec{x}$)=AS $\left[\ \vec{x}\ \right]_B$
and also T($\vec{x}$)=SB $\left[\ \vec{x}\ \right]_B$,
so that AS $\left[\ \vec{x}\ \right]_B$ =SB $\left[\ \vec{x}\ \right]_B$ for all $\vec{x}$.

Thus,

$$\text{AS=SB and A=SB}S^{-1}$$

Now we can find the standard matrix A of T:

A=SBS^{-1}
$= \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(\frac{1}{10} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \right)$
$= \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}$

Alternatively, we could use Fact 2.2.5 to construct matrix A. The point here was to explore the relationship between matrices A and B.

Fact 3.4.4
Standard matrix versus B-matrix of a linear transformation
Consider a linear transformation T from R^n to R^n and a basis B of R^n consisting of vectors $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_n}$. Let B be the B-matrix of T and let A be the standard matrix of T(such that T($\vec{x}$)=A$\vec{x}$). Then, $AS = SB$, $B = S^{-1}AS$, and $A = SBS^{-1}$, where

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & & | \end{bmatrix}$$

Definition 3.4.5 Similar matrices
Consider two n $\times$ n matrices A and B. We say that A is similar to B if there is an invertible matrix S such that

AS=SB, or B=S^{-1}AS

EXAMPLE 6

Is matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ similar to $B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$?

Solution

We are looking for a matrix S= $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that AS=SB, or

$$\begin{bmatrix} x + 2z & y + 2t \\ 4x + 3z & 4y + 3t \end{bmatrix} = \begin{bmatrix} 5x & -y \\ 5z & -t \end{bmatrix}.$$

These equations simplify to

$$z = 2x, t = -y,$$

so that any invertible matrix of the form

$$S = \begin{bmatrix} x & y \\ 2x & -y \end{bmatrix}$$

does the job. Note that det(S)=-3xy. Matrix S is invertible if det(S)≠0 (i.e.,if neither x nor y is zero).

EXAMPLE 7

Show that if matrix A is similar to B, then its power A^t is similar to B^t for all positive integers t. (That is, A^2 is similar to B^2, A^3 is similar to B^3, etc.)

Solution

We know that B=S^{-1}AS for some invertible matrix S. Now, B^t

$$= \underbrace{(S^{-1}AS)(S^{-1}AS)...(S^{-1}AS)(S^{-1}AS)}_{t-times}$$

$$= S^{-1}A^tS,$$

proving our claims. Note the cancellation of many terms of the form SS^{-1}.

Fact 3.4.6
Similarity is an equivalence relation

1. An $n \times n$ matrix A is similar to itself (Reflexivity).

2. If A is similar to B, then B is similar to A (Symmetry).

3. If A is similar to B and B is similar to C, then A is similar to C (Transitivity).

Proof
A is similar to B: $B = P^{-1}AP$
B is similar to C: $C = Q^{-1}BQ$, then

$$C = Q^{-1}BQ = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ)$$

that is, A is similar to C by matrix PQ.

Homework Exercise 3.4: 5, 6, 9, 10, 13, 14, 19, 31, 39